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A diffusive competition model with a protection zone

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Abstract

This paper is concerned with a two species diffusive competition model with a protection zone for the weak competitor. Our mathematical results imply that when the protection zone is above a certain critical patch size determined by the birth rate of the weak competitor, the weak species almost always survives, but it cannot survive when the protection zone is below the critical size and its competitor is strong enough. While this is the main feature of the model, the actual dynamical behavior of the reaction–diffusion system is more complicated. The key to reveal the main feature of the system lies in a detailed analysis of the attracting regions of its steady-state solutions. Our mathematical analysis shows that, compared with the predator–prey model discussed in [Yihong Du, Junping Shi, A diffusive predator–prey model with a protect zone, J. Differential Equations 226 (2006) 63–91], the protection zone has some essentially different effects on the fine dynamics of the competition model.

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Keywords: Reaction–diffusion system; Competition model; Spatial heterogeneity; Protection zone

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1. Introduction

The effect of heterogeneity of the environment on competition models has been considered in several recent papers. In [25,27], this problem was studied for certain competition models which may be viewed as perturbations of the degenerate limiting case where the two species are identical, and several interesting phenomena were revealed. In [11–13], another kind of degeneracy was used to capture the effects of spatial heterogeneity on the competition model, namely, the self crowding effect for one of the species is assumed to be zero (or small) on part of the habitat and essential differences of the model's behavior from the homogeneous case are revealed. In this paper, we consider a related but different situation, that is, we consider the case that one of the two species is protected from its competitor in a subregion of the habitat, called a *protection zone*, in the following sense: the protected species can leave and enter the protection zone freely but its competitor can only live outside of it.

To save or protect certain species, various protection zones are being used by the human being; the hypothetical situation to be examined here is perhaps one of the simplest. Even if the environment can be viewed as homogeneous, the introduction of a protection zone breaks the homogeneous environment for the affected species. It is important to understand the effects of such protection zones on the affected species. This paper forms part of our attempt to address this question. In a recent paper [19], a two species predator–prey model with such a protection zone for the prey was examined, and it was shown that there exists a critical patch size for the protection zone: below that size, the dynamics of the model behaves similarly to the no-protection zone case, but the dynamical behavior is fundamentally changed when the protection zone is above the critical size. As will become clear below, to understand the effect of such a protection zone on the competition model, rather different techniques are required. Moreover, our analysis suggests that the dynamics of the competition model is more complicated and the effect of the protection zone in the competition model exhibits some essential differences from that observed for the predator–prey model considered in [19].

As in [19], to focus our attention on the main features of the problem, we only consider a simplified version of the general competition model, though our techniques work as well for the general case. More precisely, we will consider the system

$$\begin{cases} u_t - \Delta u = \lambda u - u^2 - \eta b(x)uv, & x \in \Omega,\ t > 0, \\ v_t - \Delta v = \mu v - v^2 - duv, & x \in \Omega_1,\ t > 0, \\ \partial_\nu u = 0, \quad x \in \partial\Omega,\ t > 0, & \partial_\nu v = 0, \quad x \in \partial\Omega_1,\ t > 0, \\ u(x,0) = u_0(x) \geqslant 0, \quad x \in \Omega, & v(x,0) = v_0(x) \geqslant 0, \quad x \in \Omega_1, \end{cases} \tag{1.1}$$

and the corresponding elliptic problem

$$\begin{cases} -\Delta u = \lambda u - u^2 - \eta b(x)uv, & x \in \Omega, \\ -\Delta v = \mu v - v^2 - duv, & x \in \Omega_1, \\ \partial_\nu u = 0, \quad x \in \partial\Omega, & \partial_\nu v = 0, \quad x \in \partial\Omega_1. \end{cases} \tag{1.2}$$

Here Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$, Ω_0 is a subdomain of Ω such that $\overline{\Omega}_0 \subset \Omega$ and $\partial\Omega_0$ is also smooth, $\Omega_1 = \Omega \setminus \overline{\Omega}_0$. The larger region Ω is the habitat of the species u, with Ω_0 its protection zone; thus the species v can only exist in Ω_1. The Neumann boundary conditions over $\partial\Omega$ mean that the two competing species u and v live in a closed ecosystem. The Neumann boundary condition for v over $\partial\Omega_0$ represents the assumption that v

cannot go across $\partial\Omega_0$. In order to focus our attention on the effect of the protection zone on the model, we have simplified the model by assuming λ, μ, d and η being positive constants, though we could have allowed them to be positive functions of x (we could have also added different diffusion rates d_1 and d_2 for u and v, respectively). The function $b(x)$ is zero when $x \in \overline{\Omega}_0$, representing the assumption that the species u feels no competition from v in Ω_0; this also makes the interaction term in the equation for u well defined over the entire Ω. We assume that $b(x) = b > 0$ when $x \in \overline{\Omega} \setminus \overline{\Omega}_0$. We will be particularly interested in the case that η is large, which implies that the impact of competition on u is strong (so v is the stronger competitor between the two species); without a protection zone for u, it is well known that in such a case, u will die out in the long time.

Our analysis in this paper shows that the dynamical behavior of (1.1) is more complicated than that of the predator–prey system with a protection zone considered in [19]. Even in the case that η is large, there may exist more than one stable steady-state solutions. This fact makes the dynamics of the model much more difficult to understand. However, our estimates of the attracting regions of the steady-state solutions for large η always reveal a dominating stable steady-state whose attracting region is almost the entire space of the initial states. This is the key for us to capture the main feature of the dynamical behavior of the model.

We now explain in more detail our results. Similar to [19], our analysis here shows that when the birth rate of u is fixed, there is a critical patch size for the protection zone Ω_0, described by $\lambda_1^D(\Omega_0) = \lambda$, where $\lambda_1^D(\Omega_0)$ stands for the first eigenvalue of $-\Delta$ over Ω_0 under Dirichlet boundary conditions. When Ω_0 is above the critical patch size, namely $\lambda_1^D(\Omega_0) < \lambda$, then for large η, the dynamics of (1.1) largely stabilizes at a coexistence steady-state (U_η, V_η). To be more precise, we need to distinguish two subcases. If $\mu > d\lambda$, so that the semitrivial steady-states $(\lambda, 0)$ and $(0, \mu)$ are both unstable, then (U_η, V_η) is the unique positive steady-state of (1.1), and it is globally attractive to every positive solution of (1.1) (see Theorem 3.11). If $\mu < d\lambda$, so that $(0, \mu)$ is unstable but $(\lambda, 0)$ is stable, then (1.1) exhibits a bistable dynamics: apart from the stable steady-state $(\lambda, 0)$, there exists a stable positive steady-state (U_η, V_η). Moreover, $(\lambda, 0)$ only has a small attracting region in the space of initial data while (U_η, V_η) has a big attracting region $\mathcal{A}_\eta$. In fact, $\mathcal{A}_\eta$ increases with η and any given initial state (u_0, v_0) with $u_0 \not\equiv 0$ and $v_0 \not\equiv 0$ belongs to $\mathcal{A}_\eta$ when η is large enough, though for fixed $\eta > 0$ large, one can always find some initial state that does not belong to $\mathcal{A}_\eta$. As a matter of fact, it is shown that (1.1) always has an unstable positive steady-state (u_η, v_η) for large η, and v_η is always small (of the order $1/\eta$). See Theorems 3.1, 3.4 and 3.7 for details.

Therefore, for large η, (U_η, V_η) dominates the dynamics of (1.1). Moreover, as $\eta \to \infty$, U_η and V_η segregate: U_η concentrates in the protection zone Ω_0, and inside Ω_0, it is close to w_λ, which is the population of a species governed by a classical logistic law in Ω_0 with birth rate λ and hostile boundary $\partial\Omega_0$; V_η is close to its carrying capacity μ in Ω_1. The biological implication of the above fact is clear: When v is a much stronger competitor, virtually every individual of the u species is driven out of the unprotected region Ω_1, and inside the protection zone, its population will come close to what the protection zone Ω_0 can support (with birth rate λ and hostile boundary). The impact of the protection zone on the stronger species v is that its total population drops from $\mu|\Omega|$ to $\mu(|\Omega| - |\Omega_0|)$, where $|O|$ denotes the volume of O. The impact of the protection zone on the weaker species u is more profound: Without the protection zone, u is destined to extinction with a strong competitor; but when there is a protection zone whose size is larger than a certain critical size, u can survive at least inside the protection zone.

If the protection zone Ω_0 is below the critical patch size, i.e., $\lambda_1^D(\Omega_0) > \lambda$, then for large η, the population of the two competing species largely stabilizes at $(0, \mu)$. More precisely, if $\mu < d\lambda$, so that $(\lambda, 0)$ and $(0, \mu)$ are both locally stable, then (1.1) is bistable but $(0, \mu)$ is dominating when η is large. In fact, any initial state (u_0, v_0) with $v_0 \not\equiv 0$ belongs to the attracting region of $(0, \mu)$ when η is sufficiently large, though for fixed large η, there is always some initial state that does not belong to the attracting region of $(0, \mu)$; for example, there is always a positive steady-state (u_η, v_η) of (1.1) for large η, but we can show that v_η is small (of the order $1/\eta$). If $\mu > d\lambda$, the dynamics is much simpler; in such a case, for every large η, $(0, \mu)$ attracts all the positive solutions of (1.1). See Theorems 3.12, 3.13 and 3.15 for details. Biologically, these results imply that, if the protection zone is below the critical size, the weak species cannot survive when its competitor is strong enough.

In sharp contrast to the predator–prey model with a protection zone considered in [19], where it is shown that the dynamical behavior is similar to the no-protection zone case when the protection zone is below the critical patch size, here even in the small protection zone case $\lambda_1^D(\Omega_0) > \lambda$, mathematically, (1.1) exhibits a strikingly different behavior to the no-protection zone case, namely, when $\mu < d\lambda$, (1.1) always has a positive steady-state for large η. Thus, even though the attracting region of $(0, \mu)$ is large for fixed big η, it does not attract every nontrivial initial state. Without a protection zone, for large η, $(0, \mu)$ attracts every initial state (u_0, v_0) as long as $v_0 \not\equiv 0$.

A relevant biological question is the design of an optimal protection zone; we refer to [19] for some discussions.

Further related work on competition models in heterogeneous environment can be found, for example, in [2,3,14,15,26,28,32] and the references therein. Similar related problems on predator–prey models were considered in [10,17,18,20,21], and [19] examined the effect of a protection zone on a diffusive predator–prey model. However, the effect of protection zones does not seem to be considered before for diffusive competition models.

When the environment of a competition system is homogeneous, there exists a vast literature on the existence of positive steady-state solutions and their behavior, see, for example, [1,4–7,9, 16,22–24,29–31,33,34,36,37] and the references therein.

Much research and debates can be found in the literature on the establishment of marine reserves [38,39,41,42], which are protection zones where fishing activities are banned. There seems no mathematical analysis based on reaction–diffusion models for this problem, and our model in this paper does not seem to fit into the marine reserve situation, though our work might shed some light on this problem.

The rest of this paper is organized as follows. In Section 2, we collect some preliminary results, which follow easily from existing results and techniques, and will be used in Section 3, where we state and prove the main mathematical results of this paper. The main ideas and techniques are developed in Section 3.1, where the case $\lambda > \lambda_1^D(\Omega_0)$ and $\mu < d\lambda$ is analyzed, through the use of monotone dynamical systems, fixed point index theory, blowing-up techniques for elliptic equations, comparison principles for elliptic and parabolic problems, and various elliptic estimates. The remaining cases are discussed in Sections 3.2–3.4, where the proofs are usually less detailed because the main ideas and techniques are similar to those in Section 3.1.

2. Preliminaries

We collect some preliminary results here for convenience of later use. Linear eigenvalue problems will play an important role in our analysis. We denote by $\lambda_1^D(\phi, O)$ and $\lambda_1^N(\phi, O)$ the first eigenvalues of $-\Delta + \phi$ over O, with Dirichlet or Neumann boundary conditions, respectively.

We usually omit O in the notation if $O = \Omega$. If the potential function ϕ is omitted, then we understand that $\phi = 0$. We recall some well-known properties of $\lambda_1^D(\phi, O)$ and $\lambda_1^N(\phi, O)$:

1. $\lambda_1^D(\phi, O) > \lambda_1^N(\phi, O)$;
2. $\lambda_1^B(\phi_1, O) > \lambda_1^B(\phi_2, O)$ if $\phi_1 \geqslant \phi_2$ and $\phi_1 \not\equiv \phi_2$, for $B = D, N$;
3. $\lambda_1^D(\phi, O_1) \geqslant \lambda_1^D(\phi, O_2)$ if $O_1 \subset O_2$.

It is easy to see that (1.2) has two semitrivial solutions $(\lambda, 0)$ and $(0, \mu)$, and it is well known that $(\lambda, 0)$ attracts all the solutions of (1.1) with $u_0 \neq 0$ and $v_0 \equiv 0$, and $(0, \mu)$ attracts all the solutions of (1.1) with $v_0 \neq 0$ and $u_0 \equiv 0$. Moreover, standard linearization analysis shows that $(\lambda, 0)$ is linearly stable as a steady-state solution of (1.1) if $\mu < d\lambda$ and linearly unstable if $\mu > d\lambda$.

To analyze the linearized stability of $(0, \mu)$, we will need the function

$$f(\xi) := \lambda_1^N\big(b(x)\xi, \Omega\big).$$

Clearly, for any $\xi > 0$,

$$f(\xi) < \lambda_1^D\big(b(x)\xi, \Omega\big) < \lambda_1^D\big(b(x)\xi, \Omega_0\big) = \lambda_1^D(\Omega_0).$$

On the other hand, for any $\lambda \in (0, \lambda_1^D(\Omega_0))$, by Theorem 2.1 in [19], there exists a unique $\xi_0 > 0$ such that

$$f(\xi_0) = \lambda_1^N\big(b(x)\xi_0, \Omega\big) = \lambda. \tag{2.1}$$

These facts imply, by a standard linearization consideration, the following result.

Lemma 2.1. 1. *If* $\lambda \geqslant \lambda_1^D(\Omega_0)$, *then* $(0, \mu)$ *is a linearly unstable steady state of* (1.1) *for any* $\mu > 0$.

2. *If* $0 < \lambda < \lambda_1^D(\Omega_0)$, *then* $(0, \mu)$ *is linearly unstable if* $\eta < \eta_0 := \xi_0/\mu$ *and* $(0, \mu)$ *is linearly stable if* $\eta > \eta_0$.

Consider next the logistic equation

$$-\Delta u = \lambda u - u^2 - b(x)\eta u, \quad x \in \Omega, \qquad \partial_\nu u = 0, \quad x \in \partial\Omega. \tag{2.2}$$

It is well known that (2.2) has no positive solution if $\lambda \leqslant \lambda_1^N(b(x)\eta, \Omega)$, and it has a unique positive solution if $\lambda > \lambda_1^N(b(x)\eta, \Omega)$.

Let us define $W_\lambda(x)$ by

$$W_\lambda(x) = \begin{cases} 0 & \text{in } \Omega_1, \\ w_\lambda(x) & \text{in } \Omega_0, \end{cases} \tag{2.3}$$

where w_λ denotes the unique positive solution of

$$-\Delta w = \lambda w - w^2, \quad x \in \Omega_0, \qquad w = 0, \quad x \in \partial\Omega_0.$$

Treating η as a parameter, we have the following conclusions.

Lemma 2.2. 1. *If* $0 < \lambda < \lambda_1^D(\Omega_0)$, *then* (2.2) *has no positive solution for any* $\eta \geqslant \xi_0$, *and it has a unique positive solution if* $0 < \eta < \xi_0$.

2. *If* $\lambda > \lambda_1^D(\Omega_0)$, *then* (2.2) *has a unique positive solution for any* $\eta > 0$; *moreover, if we denote this solution by* z_η, *then* $\eta_1 > \eta_2 > 0$ *implies* $z_{\eta_2}(x) > z_{\eta_1}(x) > W_\lambda$, $\forall x \in \overline{\Omega}$, *and as* $\eta \to \infty$, $z_\eta \to W_\lambda$ *weakly in* $H^1(\Omega)$ *and strongly in* $C(\overline{\Omega})$.

Proof. The existence and nonexistence results follow from the well-known conclusions for logistic equations mentioned above and the definition of ξ_0. The monotonicity of z_η in η follows from a standard comparison consideration. It remains to show the last conclusion that $z_\eta \to W_\lambda$ as $\eta \to \infty$. This follows from the arguments in the proof of Propositions 3.3 and 3.2 in [19]; it is easily seen that these arguments continue to work when $m = 0$ there. $\square$

Denote by K_1, K_2 the cones of all nonnegative functions in $C(\overline{\Omega}), C(\overline{\Omega}_1)$, respectively. Denote $u \leqslant u'$ whenever $u' - u \in K_1$ and $v \leqslant v'$ whenever $v' - v \in K_2$, moreover, $u < u'$ whenever $u \leqslant u'$, $u \neq u'$ and $v < v'$ whenever $v \leqslant v'$, $v \neq v'$. Let $K = K_1 \times K_2$. It is well known that a competitive parabolic system with two species generates a monotone dynamical system (see [23,24,40]). Our situation here is slightly different from the standard case since the equations for u and v are over different spatial domains, Ω and Ω_1, respectively. However, one easily checks that the standard theory carries over to our situation. Thus, we can apply the maximum principles and the theory of monotone dynamical systems (as in, for example, [35] and [40]) to obtain the following two propositions. (We omit the proofs since they are easy modifications of well-known arguments.)

Proposition 2.3. *Suppose that* (u_1, v_1) *and* (u_2, v_2) *are the solutions of* (1.1) *with continuous nonnegative initial data* (ϕ_1, ψ_1) *and* (ϕ_2, ψ_2), *respectively. Then* $u_1(\cdot, t) \geqslant u_2(\cdot, t)$ *and* $v_2(\cdot, t) \geqslant v_1(\cdot, t)$ *for any* $t > 0$ *whenever* $\phi_1 \geqslant \phi_2$ *and* $\psi_2 \geqslant \psi_1$. *Moreover, if* $(\phi_1, \psi_1) \not\equiv (\phi_2, \psi_2)$ *and one of the following conditions is satisfied*

$$
\begin{aligned}
&(1) \quad \phi_1 \not\equiv 0, \quad \psi_1 \not\equiv 0,\\
&(2) \quad \phi_2 \not\equiv 0, \quad \psi_2 \not\equiv 0,
\end{aligned}
$$

then $u_1(x, t) > u_2(x, t)$ $\forall x \in \overline{\Omega}$ *and* $v_2(x, t) > v_1(x, t)$ $\forall x \in \overline{\Omega}_1$ *for any* $t > 0$.

Proposition 2.4. *Suppose that* (u, v) *is the unique solution of* (1.1) *with nonnegative initial data* $(\phi, \psi) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}_1)$ *satisfying, in the weak sense,*

$$
\begin{cases}
-\Delta\phi \geqslant \lambda\phi - \phi^2 - \eta b(x)\phi\psi, & x \in \Omega,\\
-\Delta\psi \leqslant \mu\psi - \psi^2 - d\phi\psi, & x \in \Omega_1,\\
\partial_\nu u = 0, \quad x \in \partial\Omega, \quad \partial_\nu v = 0, & x \in \partial\Omega_1.
\end{cases}
\tag{2.4}
$$

Then $u(\cdot, t) - u(\cdot, t') \in K_1$ *and* $v(\cdot, t') - v(\cdot, t) \in K_2$ *for any* $t' > t > 0$. *Moreover,* $(U, V) = \lim_{t\to\infty}(u(\cdot, t), v(\cdot, t))$ *exists, and* (U, V) *is a solution of* (1.2).

Proposition 2.5. *If* (U, V) *is a positive solution of* (1.2), *that is,* $U > 0$ *in* Ω *and* $V > 0$ *in* Ω_1, *then*

$$
z_{\eta\mu}(x) < U(x) < \lambda \quad \forall x \in \overline{\Omega}, \qquad V(x) < \mu \quad \forall x \in \overline{\Omega}_1.
$$

Proof. Since V satisfies

$$-\Delta V < \mu V - V^2 \quad \text{in } \Omega_1, \qquad \partial_\nu V = 0 \quad \text{on } \partial\Omega_1,$$

a simple comparison consideration shows that $0 < V < \mu$. It follows that

$$\lambda U - U^2 - \eta b(x)\mu U \leqslant (\not\equiv) -\Delta U \leqslant (\not\equiv)\, \lambda U - U^2 \quad \text{in } \Omega, \qquad \partial_\nu U = 0 \quad \text{on } \partial\Omega.$$

Therefore we can use a comparison argument to conclude that $z_{\eta\mu} < U < \lambda$ in $\overline{\Omega}$. $\square$

Proposition 2.6. (i) *Suppose* $\lambda > \lambda_1^N(\eta\mu b(x))$ *so that* $(0,\mu)$ *is unstable. Then there exists a nonnegative steady-state solution* (U,V) *with* $U(x) > 0\ \forall x \in \overline{\Omega}$ *such that any solution* (u,v) *of* (1.1) *with* $u_0, v_0 \not\equiv 0$ *satisfies*

$$\liminf_{t\to\infty} u(\cdot,t) \geqslant U, \qquad \limsup_{t\to\infty} v(\cdot,t) \leqslant V \tag{2.5}$$

uniformly in x. *Furthermore, if* $u_0 \leqslant U$, $v_0 \geqslant V$, *then* $\lim_{t\to\infty}(u(\cdot,t), v(\cdot,t)) = (U,V)$.

(ii) *Suppose* $\mu > d\lambda$ *so that* $(\lambda,0)$ *is unstable. Then there exists a nonnegative steady-state solution* (U',V') *with* $V'(x) > 0\ \forall x \in \overline{\Omega}_1$ *such that any solution* (u,v) *of* (1.1) *with* $u_0, v_0 \not\equiv 0$ *satisfies*

$$\liminf_{t\to\infty} v(\cdot,t) \geqslant V', \qquad \limsup_{t\to\infty} u(\cdot,t) \leqslant U' \tag{2.6}$$

uniformly in x. *Furthermore, if* $u_0 \geqslant U'$, $v_0 \leqslant V'$, *then* $\lim_{t\to\infty}(u(\cdot,t), v(\cdot,t)) = (U',V')$.

Proof. Consider the linearized eigenvalue problem of (1.2) at the semitrivial solution $(0,\mu)$,

$$\begin{cases} -\Delta h = \lambda h - \eta b(x)\mu h + \sigma h, & x \in \Omega, \\ -\Delta k = -\mu k - d\mu h + \sigma k, & x \in \Omega_1, \\ \partial_\nu h = 0, \quad x \in \partial\Omega, & \partial_\nu k = 0, \quad x \in \partial\Omega_1. \end{cases} \tag{2.7}$$

Since $\lambda > \lambda_1^N(\eta\mu b(x))$, there exist σ, h with $\sigma < 0$ and $h(x) > 0\ \forall x \in \overline{\Omega}$ satisfying the first equation with Neumann boundary conditions. Let $\|h\|_\infty = 1$, $\mu_\epsilon = \mu + \epsilon$ and $h_\delta = \delta h$, where ϵ, δ are positive numbers. Then there are $\epsilon_0 > 0$, $\delta_0 > 0$ such that, for all $\epsilon \leqslant \epsilon_0$ and $\delta \leqslant \delta_0$,

$$\begin{cases} -\Delta h_\delta = \lambda h_\delta - \eta b(x)\mu h_\delta + \sigma h_\delta \leqslant \lambda h_\delta - h_\delta^2 - \eta b(x)\mu_\epsilon h_\delta, & x \in \Omega, \\ -\Delta \mu_\epsilon \geqslant \mu\mu_\epsilon - \mu_\epsilon^2 - d h_\delta \mu_\epsilon, & x \in \Omega_1. \end{cases} \tag{2.8}$$

Therefore we can use Proposition 2.4 to conclude that the solution (u,v) of (1.1) with initial data (h_δ, μ_ϵ) converges to a solution $(U_{\delta,\epsilon}, V_{\delta,\epsilon})$ of (1.2) as $t \to \infty$. Similar to the proof of Proposition 2.5, we have $V_{\delta,\epsilon}(x) < \mu < \mu_\epsilon\ \forall x \in \overline{\Omega}_1$ for all $\epsilon < \epsilon_0$. It follows that

$$-\Delta U_{\delta,\epsilon} = \lambda U_{\delta,\epsilon} - U_{\delta,\epsilon}^2 - \eta b(x) U_{\delta,\epsilon} V_{\delta,\epsilon} \geqslant \lambda U_{\delta,\epsilon} - U_{\delta,\epsilon}^2 - \eta b(x)\mu U_{\delta,\epsilon}.$$

Hence $U_{\delta,\epsilon}(x) \geqslant z_{\eta\mu} > h_\delta(x)\ \forall x \in \overline{\Omega}$ for all $\delta < \delta_0$, provided that δ_0 is sufficiently small.

Let (u_i, v_i) be the solution of (1.1) with initial data $(h_{\delta_i}, \mu_{\epsilon_i})$, where $\delta_i \in (0,\delta_0)$ and $\epsilon_i \in (0,\epsilon_0)$, $i = 1,2$. The above analysis shows that (u_1,v_1) converges to $(U_{\delta_1,\epsilon_1}, V_{\delta_1,\epsilon_1})$, and that there is some $t>0$ such that $u_1(\cdot,t)\geqslant h_{\delta_2}$ and $v_1(\cdot,t)\leqslant \mu_{\epsilon_2}$. Hence from Proposition 2.3, $U_{\delta_1,\epsilon_1}\geqslant U_{\delta_2,\epsilon_2}$ and $V_{\delta_1,\epsilon_1}\leqslant V_{\delta_2,\epsilon_2}$. We can analogously show that the reversed inequalities hold. This implies that $(U_{\delta,\epsilon},V_{\delta,\epsilon})$ is independent of the choice of δ,ϵ as long as $\delta\in(0,\delta_0)$, $\epsilon\in(0,\epsilon_0)$; we will denote it by (U,V).

If (u,v) is a solution of (1.1) with initial data (u_0,v_0) satisfying $u_0\not\equiv 0$, $v_0\not\equiv 0$, then $\limsup_{t\to\infty} v(x,t)\leqslant\mu$ and hence there is some $t_0>0$ and $\epsilon>0$ such that $v(x,t_0)\leqslant\mu_\epsilon$. Since $u(x,t_0)>0$ in $\overline{\Omega}$, we can find $\delta>0$ such that $u(\cdot,t)\geqslant h_\delta$. Using Proposition 2.3, we find $\liminf_{t\to\infty} u(\cdot,t)\geqslant U$ and $\limsup_{t\to\infty} v(\cdot,t)\leqslant V$. Furthermore, if $u_0\leqslant U$, $v_0\geqslant V$, then $u(\cdot,t)\leqslant U$ and $v(\cdot,t)\geqslant V$ for all $t>0$; hence necessarily $\lim_{t\to\infty}(u(\cdot,t),v(\cdot,t))=(U,V)$.

The conclusion about (U',V') can be proved analogously, where we use the linearized eigenvalue problem of (1.2) at the solution $(\lambda,0)$ to replace (2.7). We omit the details. □

3. Main results

In this section, we will examine in detail the positive steady-state solutions and the dynamical behavior of (1.1). Our analysis will be carried out according to the following four cases:

1. $\lambda>\lambda_1^D(\Omega_0)$, $\mu<d\lambda$, namely $(\lambda,0)$ is linearly stable, $(0,\mu)$ is linearly unstable.
2. $\lambda>\lambda_1^D(\Omega_0)$, $\mu>d\lambda$, and therefore both $(\lambda,0)$ and $(0,\mu)$ are linearly unstable.
3. $\lambda<\lambda_1^D(\Omega_0)$, $\mu<d\lambda$, so $(\lambda,0)$ is linearly stable, $(0,\mu)$ is linearly unstable if $\eta<\eta_0$ and linearly stable if $\eta>\eta_0$.
4. $\lambda<\lambda_1^D(\Omega_0)$, $\mu>d\lambda$, and thus $(\lambda,0)$ is linearly unstable, $(0,\mu)$ is linearly unstable if $\eta<\eta_0$ and linearly stable if $\eta>\eta_0$.

Here $\eta_0=\xi_0/\mu$ and $\xi_0>0$ is uniquely determined by (2.1).

3.1. *Case 1: $\lambda>\lambda_1^D(\Omega_0)$, $\mu<d\lambda$*

As observed above, in this case, $(\lambda,0)$ is stable and $(0,\mu)$ is unstable. We first have the following theorem on the steady-state solutions of (1.1).

Theorem 3.1. 1. *There exists some $\eta^*>0$ such that* (1.2) *has no positive solution for $\eta<\eta^*$, has at least one positive solution for $\eta=\eta^*$ and has at least two positive solutions for $\eta>\eta^*$.*

2. *Given any $\rho\in(0,\mu)$, there exists $\hat\eta_\rho>0$ such that for $\eta>\hat\eta_\rho$,* (1.2) *has a unique positive solution (U_η,V_η) with the property $V_\eta(x)\geqslant\rho$, $\forall x\in\overline{\Omega}_1$. Moreover, (U_η,V_η) is linearly stable and converges to (W_λ,μ) in $C(\overline{\Omega})\times C(\overline{\Omega}_1)$ as $\eta\to\infty$.*

Proof. Since the proof is rather long, we break it into several steps.

Step 1: *Existence of a positive solution for large η.*

For any positive number $\rho<\mu$, consider the equation

$$-\Delta u=\lambda u-u^2-b(x)\rho\eta u,\quad x\in\Omega,\qquad \partial_\nu u=0,\quad x\in\partial\Omega. \tag{3.1}$$

By Lemma 2.2, this equation has a unique positive solution $z_{\rho\eta}$ which converges to W_λ in $C(\overline{\Omega})$ as $\eta \to \infty$. Since $W_\lambda = 0$ in Ω_1 and $\rho < \mu$, there is some η' such that $\mu\rho - \rho^2 - d\rho z_{\rho\eta} > 0$ in Ω_1 for any $\eta \geqslant \eta'$. Now we fix $\eta \geqslant \eta'$. Then $(u, v) = (z_{\rho\eta}, \rho)$ satisfies

$$\begin{cases} -\Delta u = \lambda u - u^2 - \eta b(x)uv, & x \in \Omega, \\ -\Delta v \leqslant \mu v - v^2 - duv, & x \in \Omega_1. \end{cases} \tag{3.2}$$

Consider the set $O = O_1 \times O_2$, where

$$O_1 = \{u \in C(\overline{\Omega}),\ 0 \leqslant u(x) \leqslant z_{\rho\eta}\ \forall x \in \overline{\Omega}\}, \qquad O_2 = \{v \in C(\overline{\Omega}_1),\ \rho \leqslant v(x) \leqslant \mu\ \forall x \in \overline{\Omega}_1\},$$

and the operator F on $O_1 \times O_2$ given by

$$F(u, v) = \big(F_1(u, v), F_2(u, v)\big) = \begin{pmatrix} (-\Delta + \delta)_{\Omega}^{-1}(\lambda u - u^2 - \eta b(x)uv + \delta u) \\ (-\Delta + \delta)_{\Omega_1}^{-1}(\mu v - v^2 - duv + \delta v) \end{pmatrix}^T, \tag{3.3}$$

where $\delta > 0$ is a sufficiently large constant, and $(-\Delta+\delta)_{\Omega}^{-1}$ denotes the inverse of $(-\Delta+\delta)$ over Ω with Neumann boundary conditions, $(-\Delta + \delta)_{\Omega_1}^{-1}$ denotes the inverse of $(-\Delta + \delta)$ over Ω_1 with Neumann boundary conditions. By standard elliptic regularity and the maximum principle, we see that F is a completely continuous operator from O to K. Moreover, F is order-preserving in the sense that $F_1(u, v) \geqslant F_1(u', v')$ and $F_2(u, v) \leqslant F_2(u', v')$ whenever $u \geqslant u'$ and $v \leqslant v'$. (3.2) implies that $F_1(z_{\rho\eta}, \rho) \leqslant z_{\rho\eta}$ and $F_2(z_{\rho\eta}, \rho) \geqslant \rho$. It follows that F maps O into itself. Clearly $(0, \mu)$ is a fixed point of F in O. Since $(0, \mu)$ is linearly unstable as a steady-state of (1.1), we can apply Remark 2 after Theorem 2 in [8] to conclude that F has a fixed point (u, v) in $O \setminus \{(0, \mu)\}$, which implies that (u, v) is a positive solution of (1.2), and $u(x) < z_{\rho\eta}(x)$, $\forall x \in \overline{\Omega}$, and $\rho < v(x) < \mu$, $\forall x \in \overline{\Omega}_1$.

Step 2: *Nonexistence and multiplicity of positive solutions.*

Define $\eta^* = \inf\{\eta > 0\colon$ (1.2) has a positive solution$\}$. Then $0 \leqslant \eta^* \leqslant \eta'$, and there is $\eta_n \geqslant \eta^*$ with $\eta_n \to \eta^*$ as $n \to \infty$ such that for each η_n (1.2) has a positive solution (u_n, v_n). From the equations for u_n and v_n, the boundedness of $\|u_n\|_\infty$ and $\|v_n\|_\infty$ implies that u_n and v_n are bounded in $W^{2,p}(\Omega)$ and $W^{2,p}(\Omega_1)$ ($\forall p > 1$), respectively. Hence, we can choose a subsequence of $\{(u_n, v_n)\}$ (still denoted by $\{(u_n, v_n)\}$) such that $u_n \to u_{\eta^*}$ weakly in $W^{2,p}(\Omega)$ and strongly in $C^1(\overline{\Omega})$, $v_n \to v_{\eta^*}$ weakly in $W^{2,p}(\Omega_1)$ and strongly in $C^1(\overline{\Omega}_1)$. Moreover, (u_{η^*}, v_{η^*}) is a nonnegative solution of (1.2) with $\eta = \eta^*$. We claim that (u_{η^*}, v_{η^*}) cannot be one of the three solutions $(0, 0)$, $(\lambda, 0)$ and $(0, \mu)$. In fact, from (1.2), there always holds $\lambda = \lambda_1^N(u_n + \eta b(x)v_n, \Omega)$ and $\mu = \lambda_1^N(v_n + du_n, \Omega_1)$. Since $\lambda_1^N(\phi)$ depends on $\phi \in L^\infty(\Omega)$ continuously, if $(u_n, v_n) \to (u_{\eta^*}, v_{\eta^*}) = (0, 0)$, we would deduce $\lambda = \lambda_1^N(0, \Omega) = 0$, a contradiction. If $(u_n, v_n) \to (\lambda, 0)$, we would deduce $\mu = \lambda_1^N(d\lambda, \Omega_1) = d\lambda$, contradicting our assumption that $\mu < d\lambda$. Finally if $(u_n, v_n) \to (0, \mu)$ we would deduce

$$\lambda = \lambda_1^N\big(\eta b(x)\mu, \Omega\big) < \lambda_1^D\big(\eta b(x)\mu, \Omega\big) < \lambda_1^D(\Omega_0),$$

again contradicting our assumption. Hence, (u_{η^*}, v_{η^*}) must be a positive solution. Since (1.2) has no positive solution for $\eta = 0$ (due to $\mu < d\lambda$), we necessarily have $\eta^* > 0$.

Obviously, we have

$$\begin{cases} -\Delta u_{\eta^*} \geqslant (\not\equiv)\, \lambda u_{\eta^*} - u_{\eta^*}^2 - \eta b(x) u_{\eta^*} v_{\eta^*}, & x \in \Omega, \\ -\Delta v_{\eta^*} = \mu v_{\eta^*} - v_{\eta^*}^2 - d u_{\eta^*} v_{\eta^*}, & x \in \Omega_1, \end{cases} \tag{3.4}$$

for any $\eta > \eta^*$. Hence, for any $\eta > \eta^*$, we can apply Proposition 2.4 to see that the unique solution of (1.1) with initial data (u_{η^*}, v_{η^*}) converges as $t \to \infty$ to a nonnegative solution (U_η, V_η) of (1.2), and $U_\eta(x) < u_{\eta^*}(x), \forall x \in \overline{\Omega}$ and $v_{\eta^*}(x) < V_\eta(x), \forall x \in \overline{\Omega}_1$; we have strict inequalities because of the strong maximum principle. We have $U_\eta > 0$ because of Proposition 2.6. Therefore (U_η, V_η) is a positive solution.

By Proposition 2.3, for any positive solution (u_η, v_η) of (1.2), if $u_\eta \leqslant u_{\eta^*}, v_{\eta^*} \leqslant v_\eta$ then $u_\eta \leqslant U_\eta, V_\eta \leqslant v_\eta$. For $\eta > \eta^*$ consider the set $\hat{O} = \hat{O}_1 \times \hat{O}_2$, where

$$\hat{O}_1 = \{u \in C(\overline{\Omega}),\ U_\eta(x) \leqslant u(x) \leqslant \lambda\ \forall x \in \overline{\Omega}\},$$
$$\hat{O}_2 = \{v \in C(\overline{\Omega}_1),\ 0 \leqslant v(x) \leqslant V_\eta\ \forall x \in \overline{\Omega}_1\}$$

and the operator $F : \hat{O} \to K$ defined by (3.3). We have already known that F is a completely continuous order-preserving operator. Since both (U_η, V_η) and $(\lambda, 0)$ are fixed points of F, the set $\hat{O}$ is invariant. Moreover, in view of Proposition 2.3 and the definition of (U_η, V_η), we easily see that (3.4) implies $\lim_{n\to\infty} F^n((u_{\eta^*}, v_{\eta^*})) = (U_\eta, V_\eta)$. This and the order-preserving property of F implies that, if (u, v) satisfies $U_\eta \leqslant u \leqslant u_{\eta^*}, v_{\eta^*} \leqslant v \leqslant V_\eta$, then $\lim_{n\to\infty} F^n((u, v)) = (U_\eta, V_\eta)$. On the other hand, $(\lambda, 0)$ is a linearly stable solution of (1.2). Therefore we can apply Theorem 2 of [8] to conclude that F has another fixed point in $\hat{O} \setminus \{(\lambda, 0), (U_\eta, V_\eta)\}$; that is, (1.2) has another positive solution (u'_η, v'_η) with $\lambda > u'_\eta(x) > U_\eta(x)\ \forall x \in \overline{\Omega}$ and $v'_\eta(x) < V_\eta(x)$ $\forall x \in \overline{\Omega}_1$.

Step 3: *Partial uniqueness for large η under the stability assumption.*

We have shown in Step 1 that for any ρ with $0 < \rho < \mu$, we can find a positive solution (u_η, v_η) with $v_\eta > \rho$ for sufficiently large η.

Since

$$-\Delta u_\eta = \lambda u_\eta - u_\eta^2 - b(x)\eta u_\eta v_\eta, \quad x \in \Omega, \qquad \partial_\nu u_\eta = 0, \quad x \in \partial\Omega,$$

we have

$$-\Delta u_\eta \leqslant \lambda u_\eta - u_\eta^2 - b(x)\rho\eta u_\eta, \quad x \in \Omega.$$

Hence, $z_{\eta\mu} < u_\eta \leqslant z_{\eta\rho}$. By Lemma 2.2, $z_{\eta\mu}, z_{\eta\rho} \to W_\lambda$ as $\eta \to \infty$; therefore so does u_η.

For any positive number $\rho < \mu$, we have shown that there is some η' such that $\mu\rho - \rho^2 - d\rho z_{\rho\eta} > 0$ in Ω_1 for any $\eta \geqslant \eta'$ and hence for any $\eta \geqslant \eta'$, $(u, v) = (z_{\rho\eta'}, \rho)$ satisfies

$$\begin{cases} -\Delta u \geqslant \lambda u - u^2 - \eta b(x) u v, & x \in \Omega, \\ -\Delta v \leqslant \mu v - v^2 - d u v, & x \in \Omega_1, \\ \partial_\nu u = 0, \quad x \in \partial\Omega, & \partial_\nu v = 0, \quad x \in \partial\Omega_1. \end{cases} \tag{3.5}$$

If (u_η, v_η) is a positive solution of (1.2) with $\rho < v_\eta$, then from the equation for u_η we easily deduce that $u_\eta < z_{\rho\eta'}$.

Suppose that we have proved that any positive solution (u_η, v_η) of (1.2) with $\rho < v_\eta$ and $u_\eta < z_{\rho\eta'}$ is linearly stable. Then, as above, we consider the compact order-preserving operator F on the invariant set $\tilde{O} = \tilde{O}_1 \times \tilde{O}_2$, where

$$\tilde{O}_1 = \{u \in C(\overline{\Omega}),\ 0 \leqslant u(x) \leqslant z_{\rho\eta'}(x)\ \forall x \in \overline{\Omega}\},$$
$$\tilde{O}_2 = \{v \in C(\overline{\Omega}_1),\ \rho \leqslant v(x) \leqslant \mu\ \forall x \in \overline{\Omega}_1\}.$$

Since $(0, \mu)$ is unstable and is nondegenerate, it is an isolated fixed point of F and the fixed point index $i(F, (0, \mu), \tilde{O}) = 0$. If (u_η, v_η) is any other fixed point of F in $\tilde{O}$, then by assumption, it is linearly stable and hence it is isolated and $i(F, (u_\eta, v_\eta), \tilde{O}) = 1$. The compactness of F and the isolatedness of its fixed points in $\tilde{O}$ imply that there are only finitely many fixed points in $\tilde{O}$. Since $i(F, \tilde{O}, \tilde{O}) = 1$, the additivity property of the fixed point index implies that there is a unique fixed point of F in $\tilde{O} \setminus \{(0, \mu)\}$.

Step 4: *Stability.*

It remains to show that if $(u_\eta, v_\eta) \in \tilde{O} \setminus \{(0, \mu)\}$ is a solution of (1.2), then it is linearly stable. So we consider the eigenvalue problem

$$\begin{cases} -\Delta h = \lambda h - 2u_\eta h - \eta b(x) v_\eta h - \eta b(x) u_\eta k + \sigma_\eta h, & x \in \Omega, \\ -\Delta k = \mu k - 2v_\eta k - du_\eta k - dv_\eta h + \sigma_\eta k, & x \in \Omega_1, \\ \partial_\nu h = 0, \quad x \in \partial\Omega, \quad \partial_\nu k = 0, & x \in \partial\Omega_1, \end{cases} \tag{3.6}$$

which can be transformed into an eigenvalue problem of a linear operator A on $C(\Omega) \times C(\Omega_1)$ which is compact and strongly positive with respect to the cone $K_1 \times (-K_2)$. The Krein–Rutman theorem tells us that there exist σ_η, h, k with σ_η a real number, $h(x) > 0$, $\forall x \in \overline{\Omega}$, $k(x) < 0$, $\forall x \in \overline{\Omega}_1$, satisfying (3.6). Moreover, any other eigenvalue σ satisfies $\mathrm{Re}(\sigma) > \sigma_\eta$.

Assume, for the sake of contradiction, that there are η_n with $\eta_n \to \infty$ such that $\sigma_{\eta_n} \leqslant 0$. To simplify notation, we denote $u_n = u_{\eta_n}$, $v_n = v_{\eta_n}$, $\sigma_n = \sigma_{\eta_n}$ and let (h_n, k_n) be the corresponding eigenfunction. We suppose that $\|h_n\|_\infty + \|k_n\|_\infty = 1$.

In the following, we are going to derive a contradiction. Since the arguments below are rather involved, we briefly explain the strategy here. We firstly show that $\{\eta_n\}$ has a lower bound, and hence it is a bounded sequence. We then use elliptic estimates to show that by passing to a subsequence, $k_n \to 0$ in $C^1(\overline{\Omega}_1)$. Finally we show that by passing to a further subsequence when necessary, then h_n converges to 0 uniformly in Ω. Hence we have $\|h_n\|_\infty + \|k_n\|_\infty \to 0$, contradicting our earlier assumption that $\|h_n\|_\infty + \|k_n\|_\infty = 1$.

Integrating (3.6), we have

$$\begin{cases} -\sigma_n \int\limits_{\Omega} h_n\,dx = \int\limits_{\Omega} (\lambda - 2u_n) h_n\,dx - \int\limits_{\Omega_1} \eta_n b v_n h_n\,dx - \int\limits_{\Omega_1} \eta_n b u_n k_n\,dx, \\ -\sigma_n \int\limits_{\Omega_1} k_n\,dx = \int\limits_{\Omega_1} (\mu - 2v_n - du_n) k_n\,dx - \int\limits_{\Omega_1} d v_n h_n\,dx. \end{cases} \tag{3.7}$$

Hence

$$-\sigma_n \left(\int\limits_{\Omega} h_n\,dx - \eta_n \frac{b}{d} \int\limits_{\Omega_1} k_n\,dx \right) = \int\limits_{\Omega} (\lambda - 2u_n) h_n\,dx - \eta_n \frac{b}{d} \int\limits_{\Omega_1} (\mu - 2v_n) k_n\,dx.$$

Let $M=\max\{\lambda,\mu\}$. Then

$$-\sigma_n\Big(\int_\Omega h_n\,dx-\eta_n\frac{b}{d}\int_{\Omega_1}k_n\,dx\Big)\leqslant M\Big(\int_\Omega h_n\,dx-\eta_n\frac{b}{d}\int_{\Omega_1}k_n\,dx\Big),$$

and hence $\sigma_n>-M$. Without loss of generality, we may suppose that $\sigma_n\to\sigma\leqslant 0$ as $n\to\infty$.

It is easy to see that

$$-\Delta k_n=\mu k_n-2v_nk_n-du_nk_n-dv_nh_n+\sigma_nk_n \tag{3.8}$$

is bounded in $L^\infty(\Omega_1)$. Hence, for any $p>1$, k_n is bounded in $W^{2,p}(\Omega_1)$. By passing to a subsequence, we may suppose that $\lim_{n\to\infty}k_n=k$ weakly in $W^{2,p}(\Omega_1)$ and strongly in $C^1(\overline{\Omega}_1)$.

From $-\Delta u_n=\lambda u_n-u_n^2-\eta_nb(x)u_nv_n$, we obtain

$$b\eta_n\int_{\Omega_1}u_nv_n\,dx=\int_\Omega\big(\lambda u_n-u_n^2\big)\,dx.$$

This implies that $\eta_n\int_{\Omega_1}u_n\,dx$ is bounded, since $u_n\to W_\lambda=0$ in Ω_1 and hence $v_n\to\mu$ as $n\to\infty$.

From (3.6), we also have

$$\int_\Omega|\nabla h_n|^2\,dx=\int_\Omega(\lambda-2u_n+\sigma_n)h_n^2\,dx-\eta_n\int_{\Omega_1}bv_nh_n^2\,dx-\eta_n\int_{\Omega_1}bu_nk_nh_n\,dx. \tag{3.9}$$

Since $\|h_n\|_\infty+\|k_n\|_\infty=1$ and $\eta_n\int_{\Omega_1}u_n\,dx$ is bounded, we find that $\eta_n\int_{\Omega_1}bu_nk_nh_n\,dx$ is bounded. It then follows from (3.9) that h_n is bounded in $H^1(\Omega)$ and hence there is some h such that, subject to a subsequence, $h_n\to h$ weakly in $H^1(\Omega)$ and strongly in $L^p(\Omega)$ for any $p>1$. By (3.9), $\eta_n\int_{\Omega_1}bv_nh_n^2\,dx$ is also bounded, which implies that $\int_{\Omega_1}h_n^2\,dx\to 0$ as $n\to\infty$ and hence $h=0$ almost everywhere in Ω_1.

Now we let $n\to\infty$ in (3.6) and find that k is a weak solution of the equation

$$-\Delta k=-\mu k+\sigma k\quad\text{in }\Omega_1,\qquad \partial_\nu k=0\quad\text{on }\partial\Omega_1. \tag{3.10}$$

Since $\sigma\leqslant 0$, 0 is the unique nonnegative solution of (3.10). Hence $k=0$.

We claim that for all large n, $\max_{\overline{\Omega}}h_n$ is achieved in $\overline{\Omega}_0$. Otherwise, suppose that h_n achieves its maximum value at $y_n\in\overline{\Omega}\setminus\overline{\Omega}_0$. As in [34], by the maximum principle and the Hopf boundary lemma,

$$\big[\lambda-2u_n(y_n)+\sigma_n-\eta_nbv_n(y_n)\big]h_n(y_n)-\eta_nbu_n(y_n)k_n(y_n)\geqslant 0.$$

On the other hand, we have $\eta_n\to\infty$, $v_n\to\mu$, $u_n\to 0$, $k_n\to 0$ uniformly in $\overline{\Omega}_1$ and $h_n(y_n)=\|h_n\|_\infty=1-\|k_n\|_\infty\to 1$. Hence, for all large n

$$\big[\lambda-2u_n(y_n)+\sigma_n-\eta_nbv_n(y_n)\big]h_n(y_n)-\eta_nbu_n(y_n)k_n(y_n)<0,$$

a contradiction. Our claim holds.

Let x^n be a point in $\partial\Omega_0$ such that $h_n(x^n) = \max_{\partial\Omega_0} h_n$. Without loss of generality, we may assume that $x^n \to x_0$ as $n \to \infty$ and the outer normal vector of $\partial\Omega_0$ at x_0 is $(1, 0, 0, \ldots, 0)$. We claim that $h_n(x^n) \to 0$ as $n \to \infty$. Otherwise, by passing to a subsequence, we may assume that $h_n(x^n) \geqslant \theta > 0$ for all n. Define $H_n(x) = h_n(x^n + \frac{x}{\sqrt{\eta_n}})$. Since $\overline{\Omega}_0 \subset \Omega$, there is some $c > 0$ such that H_n is well defined on the ball $B_{\sqrt{\eta_n}c}(0)$. Moreover, given any r, $\|H_n\|_{L^\infty(B_r(0))} \leqslant 1$ for sufficiently large n and we have

$$-\Delta H_n = \frac{1}{\eta_n}\left[\lambda H_n - 2u_n\left(\frac{x}{\sqrt{\eta_n}} + x^n\right)H_n - \eta_n b\left(\frac{x}{\sqrt{\eta_n}} + x^n\right)v_n\left(\frac{x}{\sqrt{\eta_n}} + x^n\right)H_n \right.$$
$$\left. - \eta_n b\left(\frac{x}{\sqrt{\eta_n}} + x^n\right)u_n\left(\frac{x}{\sqrt{\eta_n}} + x^n\right)k_n\left(\frac{x}{\sqrt{\eta_n}} + x^n\right) + \sigma_n H_n\right].$$

Since the right side of the equation is bounded in $L^\infty(B_r(0))$, H_n is bounded in $W^{2,p}(B_{r/2}(0))$ for any $p > 1$. Hence, we can suppose that $H_n \to H$ weakly in $W^{2,p}(B_{r/2}(0))$ and strongly in $C^1(\overline{B_{r/2}(0)})$. By a standard diagonal process, we can find a subsequence of H_n (still denoted by H_n) such that $H_n \to H$ in $C^1(\overline{B_R(0)})$ for any $R > 0$. Moreover, H satisfies

$$-\Delta H = -c(x)\mu H \quad \text{in } \mathbf{R}^n,\ H \geqslant 0, \tag{3.11}$$

where $c(x) = b\chi_{\{x_1 \geqslant 0\}}(x)$ (recall that $b(x) \equiv b$ in Ω_1). Furthermore, by our earlier assumption $h_n(x^n) \geqslant \theta > 0$, we have $H(0) = \lim_{n\to\infty} h_n(x^n) \geqslant \theta$.

We claim that 0 is the unique nonnegative solution of (3.11). To prove this claim, we let G_r be the unique solution of the problem

$$-\Delta G_r = -c(x)\mu G_r \quad \text{in } B_r(0), \qquad G_r = 1 \quad \text{on } \partial B_r(0).$$

Since $c(x) \geqslant 0$ and $H \leqslant G_r$ on $\partial B_r(0)$, the comparison principle shows that $H(x) \leqslant G_r(x) \leqslant 1$ in $B_r(0)$ and $G_s(x) \leqslant G_r(x)$ in $B_r(0)$ for any $s > r$. Let $G_*(x) = \lim_{r\to\infty} G_r(x)$. Then $H(x) \leqslant G_*(x) \leqslant G_r(x)$ for any $r > 0$, and $-\Delta G = -c(x)\mu G$ in $\mathbf{R}^N$. By the comparison principle, for any function G on $\mathbf{R}^N$ satisfying $-\Delta G = -c(x)\mu G$ and $G \leqslant 1$ we have $G(x) \leqslant G_r(x)$ in $B_r(0)$. It follows that $G(x) \leqslant G_*(x)$, $\forall x \in \mathbf{R}^N$. Therefore G_* is the maximal solution among all such solutions G. This implies that G_* is as symmetric as the equation allows, and hence it is a function of x_1 only. Let $f(x_1) = G_*(x)$. Then $f \in C^1(\mathbf{R}^1)$ and satisfies

$$0 \leqslant f(x_1) \leqslant 1 \quad \text{in } \mathbf{R}^1, \qquad f''(x_1) = 0 \quad \text{for } x_1 < 0, \qquad f''(x_1) = bf(x_1) \quad \text{for } x_1 > 0.$$

But $f \equiv 0$ is the unique function satisfying these conditions. This implies that $G_* = 0$ and hence $H = 0$, a contradiction to $H(0) \geqslant \theta > 0$. Hence, $h_n(x^n) \to 0$ as $n \to \infty$, as claimed.

Now, consider the equation

$$-\Delta w_n = \lambda w_n - 2u_n w_n + \sigma_n w_n, \quad x \in \Omega_0, \qquad w_n(x) = \varepsilon_n, \quad x \in \partial\Omega_0,$$

where $\varepsilon_n = h(x^n)$. Since $\lambda_1^D(2u_n - \sigma_n, \Omega_0) \to \lambda_1^D(2w_\lambda - \sigma, \Omega_0) > \lambda_1^D(w_\lambda, \Omega_0) = \lambda$, for every large n there exists a unique solution w_n satisfying the above equation and $w_n(x) > 0$, $w_n(x) \geqslant h_n(x)$ for $x \in \overline{\Omega}_0$. By standard L^p-estimates, $\{w_n\}$ is bounded in $W^{2,p}(\Omega_0)$ for all $p > 1$. Hence we can choose a subsequence, still denoted by w_n, such that $w_n \to w$ weakly in $W^{2,p}(\Omega_0)$ and strongly in $C^1(\overline{\Omega}_0)$. Moreover, w satisfies

$$-\Delta w = \lambda w - 2w_\lambda w + \sigma w, \quad x \in \Omega_0, \qquad w(x) = 0, \quad x \in \partial\Omega_0.$$

Since $\lambda_1^D(2w_\lambda - \sigma, \Omega_0) > \lambda$, the unique nonnegative solution of this equation is zero. Hence $w = 0$ and it follows from $0 \leqslant h_n \leqslant w_n$ that $h_n(x) \to 0$ uniformly for $x \in \overline{\Omega}_0$. Since the maximum of h_n over $\overline{\Omega}$ is achieved in $\overline{\Omega}_0$, this implies that $\|h_n\|_\infty \to 0$, which is a contradiction to the fact that $\|h_n\|_\infty = 1 - \|k_n\|_\infty \to 1$. This shows that our assumption $\sigma_n \leqslant 0$ can never hold for large n. Hence for sufficiently large η, (u_η, v_η) is linearly stable. $\square$

From the above proof, it is easily seen that, for sufficiently large η (depending on ρ), the positive solution (U_η, V_η) defined in Step 2 is linearly stable and it is the unique positive solution of (1.2) with the property $V_\eta \geqslant \rho$ for any given fixed $\rho \in (0, \mu)$. Moreover, in the order induced by the cone $\tilde{K} := (-K_1) \times K_2$, it is the maximal positive solution, in the sense that any other positive solution (u, v) satisfies $(u, v) \leqslant_{\tilde{K}} (U_\eta, V_\eta)$; i.e., $u \geqslant U_\eta$ in Ω, $v \leqslant V_\eta$ in Ω_1. As said before, $(\lambda, 0)$ is also linearly stable. Since F is monotone increasing in the order induced by $\tilde{K}$ and maps the order interval $[(\lambda, 0), (U_\eta, V_\eta)]_{\tilde{K}}$ into itself, by [8] we can conclude that F has a fixed point (u', v') in this order interval which is unstable, i.e., (1.2) has another positive solution which is unstable. This slightly improves the multiplicity conclusion proved in Step 2 above, since no stability conclusion is available there. In what follows, we analyze the asymptotical behavior of the positive solutions of (1.2) as $\eta \to \infty$. Note that we already knew that $(U_\eta, V_\eta) \to (W_\lambda, \mu)$ uniformly as $\eta \to \infty$, and if (u_η, v_η) is any other positive solution, then $\|v_\eta\|_\infty \to 0$.

Lemma 3.2. *Suppose that (u_η, v_η) is a positive solution of (1.2) satisfying $\|v_\eta\|_\infty \to 0$ as $\eta \to \infty$. Then we have $\|v_\eta\|_\infty = O(1/\eta)$.*

Proof. Assume, for the sake of contradiction, there is a sequence $\{\eta_n\}$ with $\eta_n \to \infty$ as $n \to \infty$ such that (1.2) with $\eta = \eta_n$ has a positive solution (u_{η_n}, v_{η_n}) satisfying $\|v_{\eta_n}\|_\infty \to 0$ and $\eta_n\|v_{\eta_n}\|_\infty \to \infty$ as $n \to \infty$. Let $(u_n, v_n) = (u_{\eta_n}, v_{\eta_n})$ and $w_n = v_n/\|v_n\|_\infty$. Then (u_n, w_n) satisfies

$$\begin{cases} -\Delta u_n = \lambda u_n - u_n^2 - \eta_n\|v_n\|_\infty b(x) u_n w_n, & x \in \Omega, \\ -\Delta w_n = \mu w_n - \|v_n\|_\infty w_n^2 - d u_n w_n, & x \in \Omega_1, \\ \partial_\nu u_n = 0, \quad x \in \partial\Omega, \quad \partial_\nu w_n = 0, & x \in \partial\Omega_1. \end{cases} \tag{3.12}$$

From the first equation and the boundedness of $\|u_n\|_\infty$, we easily see that $\int_\Omega (|\nabla u_n|^2 + u_n^2)\,dx$ is bounded. Hence u_n is bounded in $H^1(\Omega)$ and, subject to a subsequence, $u_n \to u$ weakly in $H^1(\Omega)$ and strongly in $L^p(\Omega)$ for any $p > 1$, and $u \in L^\infty(\Omega)$. From the second equation, we know that w_n is bounded in $W^{2,p}(\Omega_1)$ and hence there is a subsequence, still denoted by w_n, such that $w_n \to w$ weakly in $W^{2,p}(\Omega_1)$ and strongly in $C^1(\overline{\Omega}_1)$. It follows that $w \geqslant 0$, $\|w\|_\infty = 1$, and since $\|v_n\|_\infty \to 0$ as $n \to \infty$, w is a solution of

$$-\Delta w = \mu w - d u w, \quad x \in \Omega_1, \qquad \partial_\nu w = 0, \quad x \in \partial\Omega_1.$$

Harnack's inequality then infers that there is some $\epsilon > 0$ such that $w(x) \geqslant 2\epsilon$, $\forall x \in \overline{\Omega}_1$. From the equation for u_n we deduce

$$\lambda = \lambda_1^N\big(u_n + \eta_n\|v_n\|_\infty w_n b(x), \Omega\big) > \lambda_1^N\big(\eta_n\|v_n\|_\infty \epsilon b(x), \Omega\big) > \lambda_1^N\big(\xi_0 b(x), \Omega\big) = \lambda$$

for all large n. This contradiction completes our proof. □

If (u_η, v_η) is a positive solution of (1.2), then (u_η, w_η) with $w_\eta = \eta v_\eta$ is a positive solution of

$$\begin{cases} -\Delta u = \lambda u - u^2 - b(x)uw, & x \in \Omega, \\ -\Delta w = \mu w - \dfrac{w^2}{\eta} - duw, & x \in \Omega_1, \\ \partial_\nu u = 0, \quad x \in \partial\Omega, & \partial_\nu w = 0, \quad x \in \partial\Omega_1. \end{cases} \tag{3.13}$$

The following limit equation will be useful:

$$\begin{cases} -\Delta u = \lambda u - u^2 - b(x)uw, & x \in \Omega, \\ -\Delta w = \mu w - duw, & x \in \Omega_1, \\ \partial_\nu u = 0, \quad x \in \partial\Omega, & \partial_\nu w = 0, \quad x \in \partial\Omega_1. \end{cases} \tag{3.14}$$

For later use, contrary to elsewhere in this subsection, instead of $\lambda > \lambda_1^D(\Omega_0)$, we only assume $\lambda > 0$ in the following result.

Proposition 3.3. *Suppose $\lambda > 0$. Then problem (3.14) has no positive solution when $\mu \geqslant d\lambda$, and it has at least one positive solution when $\mu < d\lambda$.*

Proof. Suppose $\mu \geqslant d\lambda$. If there is a positive solution (u, w), then it is easy to see that $u < \lambda$, and hence $\mu \geqslant d\lambda = \lambda_1^N(d\lambda, \Omega_1) > \lambda_1^N(du, \Omega_1)$. But w is positive and the second equation in (3.14) implies that $\mu = \lambda_1^N(du, \Omega_1)$, a contradiction. Hence there is no positive solution when $\mu \geqslant d\lambda$.

Suppose $\mu < d\lambda$. Consider the problem

$$\begin{cases} -\Delta u = \lambda u - u^2 - b(x)uw, & x \in \Omega, \\ -\Delta w = \mu w - tduw, & x \in \Omega_1, \\ \partial_\nu u = 0, \quad x \in \partial\Omega, & \partial_\nu w = 0, \quad x \in \partial\Omega_1, \end{cases} \tag{3.15}$$

with parameter $t \in [0, 1]$.

We claim there is a constant c such that if (u_t, w_t) is a positive solution of (3.15), then $\|w_t\|_\infty < c$. Otherwise, we can find a sequence $(u_{t_n},, v_{t_n})$ such that $\|w_{t_n}\|_\infty \to \infty$ and $t_n \to t_0$. Let $\hat{w}_n = w_{t_n}/\|w_{t_n}\|_\infty$. Then $-\Delta\hat{w}_n = \mu\hat{w}_n - t_n d u_{t_n}\hat{w}_n$. Since $\|u_{t_n}\|_\infty, \|\hat{w}_n\|_\infty$ are bounded, $\hat{w}_n$ is bounded in $W^{2,p}(\Omega_1)$ for any $p > 1$. We may suppose $\hat{w}_n \to w$ weakly in $W^{2,p}(\Omega_1)$ and strongly in $C^1(\overline{\Omega}_1)$, $u_{t_n} \to u$ weakly in $H^1(\Omega)$ and strongly in $L^p(\Omega)$. Then w satisfies $-\Delta w = \mu w - t_0 duw$. Since $(\mu - t_0 du) \in L^\infty$, Harnack's inequality implies there is some $\epsilon > 0$ such that $w(x) > \epsilon$ for all $x \in \overline{\Omega}_1$. Thus for all large n, $\hat{w}_n(x) > \epsilon$ in $\overline{\Omega}_1$. From the equation for u_{t_n}, we obtain

$$\lambda = \lambda_1^N\big(u_{t_n} + \|w_n\|_\infty\hat{w}_n(x)b(x), \Omega\big) > \lambda_1^N\big(\|w_n\|_\infty\epsilon b(x), \Omega\big) > \lambda_1^N\big(\xi_0 b(x), \Omega\big) = \lambda$$

for all large n. This contradiction shows that our claim holds.

Define $A = A_1 \times A_2$, where

$$A_1 = \{u \in C(\overline{\Omega}),\ 0 \leqslant u(x) < 2\lambda,\ \forall x \in \overline{\Omega}\},$$
$$A_2 = \{u \in C(\overline{\Omega}_1),\ 0 \leqslant w(x) < c,\ \forall x \in \overline{\Omega}_1\}.$$

For large $\delta > 0$, consider the operator

$$F_t(u, w) = F(t, u, w) = \begin{pmatrix} (-\Delta + \delta)_{\Omega}^{-1}(\lambda u - u^2 - b(x)uw + \delta u) \\ (-\Delta + \delta)_{\Omega_1}^{-1}(\mu w - tduw + \delta w) \end{pmatrix}^T \tag{3.16}$$

on A. As before, F_t maps A into the positive cone K and is completely continuous. By the properties of the fixed point index and our a priori bound established above, we know that $i(F_1, A, K) = i(F_0, A, K)$. On the other hand, solving Eq. (3.15) with $t = 0$, we find that F_0 has in A exactly two fixed points $(0, 0)$ and $(\lambda, 0)$, and both are linearly unstable. Hence $i(F_1, A, K) = i(F_0, A, K) = i(F_0, (0, 0), K) + i(F_0, (\lambda, 0), K) = 0$. On the other hand, $(0, 0)$, $(\lambda, 0)$ and $(0, \mu)$ are fixed points of F_1 in A, and $(0, 0)$, $(0, \mu)$ are linearly unstable, $(\lambda, 0)$ is linearly stable. This implies that $i(F_1, (0, 0), K) = i(F_1, (0, \mu), K) = 0$, $i(F_1, (\lambda, 0), K) = 1$. Hence we may use the additivity property of the fixed point index to conclude that F_1 has at least one more fixed point in A, which is necessarily a positive solution of (3.14). □

Let S denote the set of positive solutions of (3.14). From the proof of Proposition 3.3, we know that $S \subset A$. By standard elliptic regularity theory, one sees that S is precompact. It is easily checked that $(0, 0)$ and $(\lambda, 0)$ are nondegenerate solutions of (3.14), and hence they are isolated solutions; thus S is a compact set in $C(\overline{\Omega}) \times C(\overline{\Omega}_1)$. We are now able to obtain a better understanding of the set of positive solutions of (1.2) for large η.

Theorem 3.4. *Given $\epsilon > 0$, there exists $\eta_\epsilon > 0$ such that for any $\eta > \eta_\epsilon$, if (u_η, v_η) is a positive solution of* (1.2) *that is different from (U_η, V_η), then $\mathbf{d}((u_\eta, \eta v_\eta), S) < \epsilon$, where $\mathbf{d}$ is the distance function in $C(\overline{\Omega}) \times C(\overline{\Omega}_1)$.*

Proof. It suffices to show that if $\eta_n \to \infty$ and if (u_n, v_n) is a positive solution of (1.2) with $\eta = \eta_n$ that is different from (U_{η_n}, V_{η_n}), then, subject to a subsequence, $(u_n, \eta_n v_n)$ converges to a positive solution of (3.14). By Lemma 3.2, $w_n := \eta_n v_n$ is bounded in $L^\infty(\Omega_1)$. From the equations satisfied by u_n and w_n, we easily see that they are bounded in $W^{2,p}(\Omega)$ and $W^{2,p}(\Omega_1)$ ($\forall p > 1$), respectively. Therefore by passing to a subsequence we may assume that $u_n \to u$ in $C^1(\overline{\Omega})$ and $w_n \to w$ in $C^1(\overline{\Omega}_1)$. It is easily seen that (u, w) is a nonnegative solution of (3.14).

It remains to show that (u, w) is a positive solution. From the equations for u_n and w_n, we find

$$\lambda = \lambda_1^N(u_n + b(x)w_n, \Omega), \qquad \mu = \lambda_1^N(w_n/\eta_n + du_n, \Omega_1).$$

If (u, w) is not a positive solution of (3.14), then either $(u, w) = (0, 0)$ or $(u, w) = (\lambda, 0)$. In the case $(u, w) = (0, 0)$, we deduce

$$\lambda = \lambda_1^N(u_n + b(x)w_n, \Omega) \to \lambda_1^N(0, \Omega) = 0,$$

contradicting the assumption that $\lambda > 0$. If $(u, w) = (\lambda, 0)$, then

$$\mu = \lambda_1^N(w_n/\eta_n + du_n, \Omega_1) \to \lambda_1^N(d\lambda, \Omega_1) = d\lambda,$$

contradicting the assumption that $\mu < d\lambda$. Therefore (u, w) is a positive solution, and the proof is complete. $\square$

Now, we study the dynamics of (1.1). We have proved that for any $\rho > 0$, there is $\hat{\eta}_\rho$ such that when $\eta > \hat{\eta}_\rho$, (U_η, V_η) is the unique positive steady state of (1.1) satisfying $V_\eta \geqslant \rho$, and it is linearly stable. Let $\mathcal{A}_\eta$ be the attracting region of (U_η, V_η) in K, that is, $\mathcal{A}_\eta$ consists of all $(u_0, v_0) \in K$ such that the solution of (1.1) with initial data (u_0, v_0) satisfies $\lim_{t\to\infty} u(x,t) = U_\eta(x)$ uniformly for $x \in \overline{\Omega}$ and $\lim_{t\to\infty} v(x,t) = V_\eta(x)$ uniformly for $x \in \overline{\Omega}_1$. We have

Lemma 3.5. *For any $\eta > \hat{\eta}_\rho$, if $(\phi_0, \psi_0) \in \mathcal{A}_\eta$, then any $(\phi, \psi) \in K$ with $0 \not\equiv \phi \leqslant \phi_0$ and $\psi \geqslant \psi_0$ also belongs to $\mathcal{A}_\eta$.*

Proof. Indeed, let (U, V) be given by Proposition 2.6; then $U_\eta \geqslant U$, $V_\eta \leqslant V$ and hence, by Theorem 3.1, $(U, V) = (U_\eta, V_\eta)$ for all large η. For such η, if (u, v) and (u^0, v^0) are the solutions of (1.1) with initial data (ϕ, ψ) and (ϕ_0, ψ_0), respectively, then by Proposition 2.3 we have $u(x,t) \leqslant u^0(x,t)$ and $v(x,t) \geqslant v^0(x,t)$ for all $t > 0$. It follows that

$$\limsup_{t\to\infty} u(x,t) \leqslant \lim_{t\to\infty} u^0(x,t) = U_\eta(x), \qquad \liminf_{t\to\infty} v(x,t) \geqslant \lim_{t\to\infty} v^0(x,t) = V_\eta(x).$$

On the other hand, by Proposition 2.6, we have

$$\liminf_{t\to\infty} u(x,t) \geqslant U(x) = U_\eta(x), \qquad \limsup_{t\to\infty} v(x,t) \leqslant V(x) = V_\eta(x).$$

Therefore, $(u, v) \to (U_\eta, V_\eta)$ as $t \to \infty$, i.e., $(\phi, \psi) \in \mathcal{A}_\eta$. $\square$

Proposition 3.6. *For $\eta_1 > \eta_2 > \hat{\eta}_\rho$, $\mathcal{A}_{\eta_1} \supset \mathcal{A}_{\eta_2}$.*

Proof. For $\eta_1 > \eta_2$, (U_{η_2}, V_{η_2}) satisfies

$$\begin{cases} -\Delta U_{\eta_2} \geqslant \lambda U_{\eta_2} - U_{\eta_2}^2 - \eta_1 b(x) U_{\eta_2} V_{\eta_2}, & x \in \Omega, \\ -\Delta V_{\eta_2} = \mu V_{\eta_2} - V_{\eta_2}^2 - d U_{\eta_2} V_{\eta_2}, & x \in \Omega_1. \end{cases} \tag{3.17}$$

Propositions 2.4 and 2.6 imply that the solution (u, v) of (1.1) with initial data (U_{η_2}, V_{η_2}) and $\eta = \eta_1$ converges to a positive solution of (1.2) with $\eta = \eta_1$ as $t \to \infty$. Since (U_{η_1}, V_{η_1}) is the unique positive solution of (1.2) with $\eta = \eta_1$ that satisfies $U_{\eta_1} \leqslant U_{\eta_2}$, $\rho \leqslant V_{\eta_2} < V_{\eta_1}$, (u, v) must converge to (U_{η_1}, V_{η_1}). Hence $(U_{\eta_2}, V_{\eta_2}) \in \mathcal{A}_{\eta_1}$. Since (U_{η_1}, V_{η_1}) is linearly stable, $\mathcal{A}_{\eta_1}$ contains a neighborhood B of (U_{η_2}, V_{η_2}) in K. Particularly, there is some $(\alpha, \beta) \in \mathcal{A}_{\eta_1}$ with $\alpha(x) > U_{\eta_2}(x)$ uniformly for $x \in \overline{\Omega}$ and $0 < \beta(x) < V_{\eta_2}(x)$ uniformly for $x \in \overline{\Omega}_1$. If $(u_0, v_0) \in \mathcal{A}_{\eta_2}$, then there is some t_0 such that $(u^2(\cdot, t_0), v^2(\cdot, t_0)) \geqslant_{\tilde{K}} (\alpha, \beta)$, where (u^2, v^2) is the solution of (1.1) with initial data (u_0, v_0) and $\eta = \eta_2$. Moreover, by a standard comparison consideration, $(u^1(\cdot, t_0), v^1(\cdot, t_0)) \geqslant_{\tilde{K}} (u^2(\cdot, t_0), v^2(\cdot, t_0)) \geqslant_{\tilde{K}} (\alpha, \beta)$, where (u^1, v^1) is the solution of (1.1) with initial data (u_0, v_0) and $\eta = \eta_1$. This implies that $(u^1(\cdot, t_0), v^1(\cdot, t_0)) \in \mathcal{A}_{\eta_1}$ by Lemma 3.5. It follows that $(u_0, v_0) \in \mathcal{A}_{\eta_1}$. The proof is complete. $\square$

Furthermore, we have the following theorem.

Theorem 3.7. *For any $(u_0, v_0) \in K$ with $u_0 \not\equiv 0$ and $v_0 \not\equiv 0$, there is some $\tilde{\eta}$ such that $(u_0, v_0) \in \mathcal{A}_\eta$ whenever $\eta > \tilde{\eta}$.*

To prove this theorem, we need some preparations. Recall that z_η is the unique positive solution of (2.2). Let $\tilde{u}_\eta(x, t, u_0)$ denote the unique solution of

$$\begin{cases} u_t - \Delta u = \lambda u - u^2 - \eta b(x)u, & x \in \Omega,\ t > 0, \\ \partial_\nu u = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) \geqslant 0, \quad u_0(x) \not\equiv 0, & x \in \Omega. \end{cases} \tag{3.18}$$

Lemma 3.8. *Given positive constants $\eta > \theta$, there is a constant $T_\eta = T_\eta(\theta)$ such that $\tilde{u}_\eta(x, t, u_0) \leqslant z_\theta(x)$ for any $x \in \overline{\Omega}$ and $t \geqslant T_\eta$. Moreover, T_η is nonincreasing in η.*

Proof. For any $\eta > \theta$, by a simple comparison consideration, $z_\eta(x) < z_\theta(x)$ in $\overline{\Omega}$. By well-known conclusions about the logistic equation, $\tilde{u}_\eta(\cdot, t, u_0) \to z_\eta$ as $t \to \infty$. Hence there is some T_η such that $\tilde{u}_\eta(x, t, u_0) \leqslant z_\theta(x)$ for any $x \in \overline{\Omega}$ and $t \geqslant T_\eta$. On the other hand, for any $\eta_1 > \eta_2$,

$$\partial_t \tilde{u}_{\eta_2} - \Delta \tilde{u}_{\eta_2} = \lambda \tilde{u}_{\eta_2} - \tilde{u}_{\eta_2}^2 - \eta_2 b(x)\tilde{u}_{\eta_2} \geqslant \lambda \tilde{u}_{\eta_2} - \tilde{u}_{\eta_2}^2 - \eta_1 b(x)\tilde{u}_{\eta_2}, \quad x \in \Omega,\ t > 0.$$

Hence $\tilde{u}_{\eta_1}(\cdot, t, u_0) \leqslant \tilde{u}_{\eta_2}(\cdot, t, u_0)$. This implies that T_η can be chosen to be nonincreasing in η. □

Since the solution z_η of (2.2) converges to W_λ (see (2.3) for its definition) uniformly as $\eta \to \infty$, given any $\epsilon > 0$, there is some $C > 0$ such that for any $\eta \geqslant C$, $z_\eta \leqslant \epsilon$ in $\overline{\Omega}_1$. Fix $\epsilon \in (0, \mu/d)$, we have

Lemma 3.9. *Let $(u(x,t), v(x,t))$ be the solution of* (1.1) *with $u(\cdot, 0) = z_C$, $v(\cdot, 0) = C/\eta$. Then there is some η_1^* such that for any $\eta > \eta_1^*$, $\lim_{t\to\infty} u(x,t) = U_\eta(x)$ uniformly for $x \in \overline{\Omega}$ and $\lim_{t\to\infty} v(x,t) = V_\eta(x)$ uniformly for $x \in \overline{\Omega}_1$.*

Proof. Due to our choice of ϵ, it is easy to see that for sufficiently large $\eta > C$, $\mu - C/\eta - d\epsilon > 0$ and therefore $u = z_C$, $v = C/\eta$ satisfy

$$\begin{cases} -\Delta u = \lambda u - u^2 - \eta b(x)uv, & x \in \Omega, \\ -\Delta v \leqslant \mu v - v^2 - duv, & x \in \Omega_1, \\ \partial_\nu u = 0, \quad x \in \partial\Omega, & \partial_\nu v = 0, \quad x \in \partial\Omega_1. \end{cases} \tag{3.19}$$

By Propositions 2.4 and 2.6, there is some positive solution (u_η, v_η) of (1.2) such that $\lim_{t\to\infty} u(x,t) = u_\eta(x)$ uniformly for $x \in \overline{\Omega}$ and $\lim_{t\to\infty} v(x,t) = v_\eta(x)$ uniformly for $x \in \overline{\Omega}_1$, where $(u(x,t), v(x,t))$ is the solution of (1.1) with initial data $(z_C, C/\eta)$. Moreover, $u(x,t) \leqslant u(x,0) = z_C(x) \leqslant \epsilon$ in $\overline{\Omega}_1$. It follows that $v_\eta(x) \geqslant \mu - d\epsilon > 0$. By Theorem 3.1, we know that $(u_\eta, v_\eta) = (U_\eta, V_\eta)$ provided that η is large enough. This completes the proof. □

Proof of Theorem 3.7. Using Lemmas 3.5 and 3.9, we only need to prove that for any $(u_0, v_0) \in K$ satisfying $u_0 \not\equiv 0$ and $v_0 \not\equiv 0$, there is a constant η_0^* such that for any $\eta > \eta_0^*$, there is some positive number M such that $u(\cdot, M) \leqslant z_C$ and $v(\cdot, M) \geqslant C/\eta$, where (u, v) is the solution of (1.1) with initial data (u_0, v_0).

Choose a constant α such that $\alpha > \max\{\lambda, \|u_0\|_\infty\}$. Then it is easily seen that $u(x,t) \leqslant \alpha$ for all $t > 0$. It follows that

$$v_t = \Delta v + \mu v - v^2 - duv \geqslant \Delta v + \mu v - v^2 - d\alpha v.$$

Hence $v \geqslant v^0$, where v^0 denotes the unique solution of

$$\begin{cases} v_t = \Delta v + \mu v - v^2 - d\alpha v, & x \in \Omega_1,\ t > 0, \\ \partial_\nu v = 0, \quad x \in \partial\Omega_1,\ t > 0, & v(x,0) = v_0(x), \quad x \in \Omega_1. \end{cases}$$

By the strong maximum principle we have $v^0(x,t) > 0$ for $x \in \overline{\Omega}_1$ and $t > 0$. Therefore, for each $\eta > [\min_{\overline{\Omega}_1} v(\cdot, 1)]^{-2}$, there is a positive number $\tau_\eta > 1$ such that $v^0(x,t) \geqslant 1/\sqrt{\eta}$ for $t \in [1, \tau_\eta]$, and we may choose τ_η such that $\tau_\eta \to \infty$ as $\eta \to \infty$.

Let η_2^* satisfy $\sqrt{\eta_2^*} > C$. Then by Lemma 3.8 there is some $M > 1$ such that for any $\eta > \eta_2^*$, $\tilde{u}_{\sqrt{\eta}}(x, M-1, \alpha) \leqslant z_C(x)$ in $\overline{\Omega}$.

We then choose η_0^* sufficiently large such that $\tau_\eta > M$ and $1/\sqrt{\eta} > C/\eta$ whenever $\eta \geqslant \eta_0^*$. Then for any $\eta \geqslant \eta_0^*$ and $t \in [1, M]$, $v(x,t) \geqslant v^0(x,t) \geqslant 1/\sqrt{\eta}$ and $u(x,t)$ satisfies, for $t \in [1, M]$,

$$u_t - \Delta u = \lambda u - u^2 - \eta b(x)uv \leqslant \lambda u - u^2 - \sqrt{\eta}b(x)u.$$

Since $u(x,1) \leqslant \alpha$, it follows that $u(x,t) \leqslant \tilde{u}_{\sqrt{\eta}}(x, t-1, \alpha)$ for $t \in [1, M]$, $x \in \overline{\Omega}$. In particular, $u(x, M) \leqslant \tilde{u}_{\sqrt{\eta}}(x, M-1, \alpha) \leqslant z_C(x)$ for any $x \in \overline{\Omega}$. Moreover, we have $v(x, M) \geqslant 1/\sqrt{\eta} > C/\eta$. The proof is complete. □

Remark 3.10. The above arguments actually prove a stronger result: For any $(u_0, v_0) \in K$ with $u_0 \not\equiv 0$ and $v_0 \not\equiv 0$, there is some $\tilde{\eta}$ such that $(\phi, \psi) \in \mathcal{A}_\eta$ whenever $\eta > \tilde{\eta}$ and $(\phi, \psi) \in K$, $0 \not\equiv \phi \leqslant u_0$, $\psi \geqslant v_0$.

3.2. Case 2: $\lambda > \lambda_1^D(\Omega_0)$, $\mu > d\lambda$

In this case, we know the semitrivial solutions $(\lambda, 0)$ and $(0, \mu)$ are both nondegenerate and unstable. Then as before we can transform (1.2) into a fixed point problem for some operator A which maps the order interval $[(\lambda, 0), (0, \mu)]_{\tilde{K}}$ into itself and is order-preserving. Since $(\lambda, 0)$ and $(0, \mu)$ are nondegenerate and linearly unstable, it is well known that A must have a stable fixed point in $[(\lambda, 0), (0, \mu)]_{\tilde{K}} \setminus \{(\lambda, 0), (0, \mu)\}$ (see [8]). Since $(0, 0)$ is unstable, it follows that (1.2) has at least one stable positive solution for any $\eta > 0$.

When η is large, we have the following much stronger result.

Theorem 3.11. *For sufficiently large η,* (1.2) *has a unique positive solution (U_η, V_η), which converges to (W_λ, μ) in $C(\overline{\Omega}) \times C(\overline{\Omega}_1)$ as $\eta \to \infty$. Moreover, it is globally attracting, namely, if $(u(x,t), v(x,t))$ is a solution of* (1.1) *with $u(x,0) \not\equiv 0$, $v(x,0) \not\equiv 0$, then* $\lim_{t\to\infty} u(x,t) = U_\eta(x)$ *uniformly for $x \in \overline{\Omega}$ and* $\lim_{t\to\infty} v(x,t) = V_\eta(x)$ *uniformly for $x \in \overline{\Omega}_1$.*

Proof. If (u_η, v_η) is a positive solution of (1.2), then we easily see that $0 < u_\eta < \lambda$ and hence

$$-\Delta v_\eta > \mu v_\eta - v_\eta^2 - d\lambda v_\eta.$$

It follows that $v_\eta > \rho_0 := \mu - d\lambda > 0$. We can then use the equation for u_η to deduce, as before, $z_{\eta\mu} \leqslant u_\eta \leqslant z_{\eta\rho_0}$. It then follows that $(u_\eta, v_\eta) \to (W_\lambda, \mu)$ in $C(\overline{\Omega}) \times C(\overline{\Omega}_1)$ as $\eta \to \infty$.

Let $(u, v) = (\lambda, \rho)$ with ρ a positive constant satisfying $\rho \leqslant \mu - d\lambda$. Then, for any $\eta > 0$,

$$\begin{cases} -\Delta u \geqslant \lambda u - u^2 - \eta b(x)uv, & x \in \Omega, \\ -\Delta v \leqslant \mu v - v^2 - duv, & x \in \Omega_1, \\ \partial_\nu u = 0, \quad x \in \partial\Omega, & \partial_\nu v = 0, \quad x \in \partial\Omega_1. \end{cases}$$

We can now repeat the arguments in Steps 3 and 4 of the proof of Theorem 3.1, with $z_{\rho\eta'}$ there replaced by λ, to conclude that any positive solution of (1.2) is linearly stable and hence there is a unique positive solution when η is sufficiently large.

Let (U_η, V_η) denote the unique positive solution of (1.2) with large η. Then for such η, applying Proposition 2.6 we obtain

$$0 < U \leqslant U_\eta \leqslant U', \qquad 0 < V' \leqslant V_\eta \leqslant V,$$

where (U, V) and (U', V') are nonnegative solutions of (1.2) as given in Proposition 2.6. The above inequalities imply $0 < U < \lambda$ and $0 < V' < \mu$. Therefore (U, V) and (U', V') are positive solutions, and by uniqueness,

$$(U, V) = (U', V') = (U_\eta, V_\eta).$$

Now it follows from (2.5) and (2.6) that if $(u(x,t), v(x,t))$ is a solution of (1.1) with $u_0 \not\equiv 0$, $v_0 \not\equiv 0$, then $\lim_{t\to\infty} u(x,t) = U_\eta(x)$ uniformly for $x \in \overline{\Omega}$ and $\lim_{t\to\infty} v(x,t) = V_\eta(x)$ uniformly for $x \in \overline{\Omega}_1$. $\square$

3.3. *Case 3: $\lambda < \lambda_1^D(\Omega_0)$, $\mu < d\lambda$*

In this case, $(\lambda, 0)$ is linearly stable, $(0, \mu)$ is unstable for $\eta < \eta_0$ and is linearly stable for $\eta > \eta_0$. About the steady-state solutions, we have the following result.

Theorem 3.12. *For $\eta > \eta_0$,* (1.2) *has an unstable positive solution $(\bar{U}_\eta, \bar{V}_\eta)$ and $\bar{V}_\eta \to 0$ as $\eta \to \infty$ in $C(\overline{\Omega}_1)$. Moreover, for any $\epsilon > 0$, there exists $\eta_\epsilon > 0$ such that if (u_η, v_η) is a positive solution of* (1.2) *with $\eta > \eta_\epsilon$, then $\mathbf{d}((u_\eta, \eta v_\eta), S) < \epsilon$, where S is the set of all positive solutions of* (3.14)*, and $\mathbf{d}$ is the distance function in $C(\overline{\Omega}) \times C(\overline{\Omega}_1)$.*

Proof. The existence of an unstable positive solution follows from a standard consideration. Indeed, we may transform (1.2) into an equivalent fixed point problem of a completely continuous operator A which maps the order interval $[(\lambda, 0), (0, \mu)]_{\tilde{K}}$ into itself and is order-preserving. Since $(\lambda, 0)$ and $(0, \mu)$ are both linearly stable and $(0, 0)$ is linearly unstable, it is well known (by a count of the fixed point indices) that A has a fixed point, say (u, v), in $[(\lambda, 0), (0, \mu)]_{\tilde{K}} \setminus \{(\lambda, 0), (0, \mu), (0, 0)\}$, which must be a positive solution of (1.2). If (u, v) is stable, then it is well known (e.g. [8]) that A must have an unstable fixed point in $[(u, v), (0, \mu)]_{\tilde{K}}$, which must be an unstable positive of solution of (1.2). Therefore (1.2) always has an unstable positive solution.

Suppose now $\eta_n \to \infty$ and (u_n, v_n) is a positive solution of (1.2) with $\eta = \eta_n$. Firstly we claim that $\|v_n\|_\infty \to 0$. Since $0 < u_n < \lambda$ and $0 < v_n < \mu$, from the equation for v_n we find that $\{v_n\}$ is bounded in $W^{2,p}(\Omega_1)$ for any $p > 1$. From the equation for u_n we easily see that $\int_\Omega (|\nabla u_n|^2 + u_n^2)\,dx$ is bounded, and hence $\{u_n\}$ is bounded in $H^1(\Omega)$. Therefore, by passing to a subsequence, we may assume that $u_n \to u$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$, $v_n \to v$ weakly in $W^{2,p}(\Omega_1)$ and strongly in $C^1(\overline{\Omega}_1)$. Moreover,

$$-\Delta v = (\mu - v - du)v \quad \text{in } \Omega_1, \qquad \partial_\nu v = 0 \quad \text{on } \partial\Omega_1.$$

Since $(\mu - v - du) \in L^\infty(\Omega_1)$, by the Harnack inequality, either $v \equiv 0$ or $v > 0$ in $\overline{\Omega}_1$. In the later case, we deduce that $v_n \geqslant \epsilon > 0$ for all large n and all $x \in \Omega_1$. It follows that

$$\lambda = \lambda_1^N\big(u_n + \eta_n v_n b(x), \Omega\big) > \lambda_1^N\big(\eta_n \epsilon b(x), \Omega\big) > \lambda_1^N\big(\xi_0 b(x), \Omega\big) = \lambda$$

for all large n. This contradiction shows that we must have $v \equiv 0$, which implies that the entire original sequence v_n converges to 0 uniformly in Ω_1 as $n \to \infty$. This proves our claim.

Secondly we claim that $\{\eta_n \|v_n\|_\infty\}$ is bounded. Arguing indirectly, we assume that this sequence is unbounded; by passing to a subsequence, we may assume that $\eta_n \|v_n\|_\infty \to \infty$. Let $w_n = v_n / \|v_n\|_\infty$; then (u_n, w_n) satisfies (3.12). As in the proof of Lemma 3.2, we find that, subject to a subsequence, $u_n \to u$ weakly in $H^1(\Omega)$ and strongly in $L^p(\Omega)$, $w_n \to w$ weakly in $W^{2,p}(\Omega_1)$ and strongly in $C^1(\overline{\Omega}_1)$, $\forall p > 1$. Moreover, w is a nonnegative solution to

$$-\Delta w = \mu w - duw \quad \text{in } \Omega_1, \qquad \partial_\nu w = 0 \quad \text{on } \partial\Omega_1.$$

Since $\|w\|_\infty = 1$, by the Harnack inequality there exists some $\epsilon > 0$ such that $w(x) \geqslant 2\epsilon$ in $\overline{\Omega}_1$. Therefore $w_n \geqslant \epsilon$ for all large n. Now from the equation for u_n we deduce

$$\lambda = \lambda_1^N\big(u_n + \eta_n \|v\|_n b(x) w_n, \Omega\big) > \lambda_1^N\big(\eta_n \|v_n\|_\infty \epsilon b(x), \Omega\big) > \lambda_1^N\big(\xi_0 b(x), \Omega\big) = \lambda$$

for all large n. This contradiction shows that our claim holds.

Finally we let $\tilde{w}_n = \eta_n v_n$. Then $(u_n, \tilde{w}_n)$ is a positive solution of (3.13) with $\eta = \eta_n$. Since $\|u_n\|_\infty$ and $\|\tilde{w}_n\|_\infty$ are both bounded, from the equations for u_n and $\tilde{w}_n$, we easily see that $\{u_n\}$ and $\{\tilde{w}_n\}$ are bounded sequences in $W^{2,p}(\Omega)$ and $W^{2,p}(\Omega_1)$ ($\forall p > 1$), respectively. Hence by passing to a subsequence, we may assume $u_n \to u$ and $\tilde{w}_m \to \tilde{w}$ weakly in the $W^{2,p}$ norm and strongly in the C^1 norm. Moreover, $(u, \tilde{w})$ is a nonnegative solution of (3.14). The argument near the end of the proof of Theorem 3.4 shows that $(u, \tilde{w})$ is a positive solution of (3.16). This completes the proof. □

Let $\mathcal{B}_\eta \subset K$ be the attracting region of $(0, \mu)$. We have the following result.

Theorem 3.13. *For any $(u_0, v_0) \in K$ with $u_0 \not\equiv 0$, $v_0 \not\equiv 0$, there is some $\tilde{\eta}$ such that $(u_0, v_0) \in \mathcal{B}_\eta$ whenever $\eta > \tilde{\eta}$.*

Proof. Choose $\xi_1 > \xi_0$ so that

$$\lambda = \lambda_1^N\big(\xi_0 b(x), \Omega\big) < \lambda_1^N\big(\xi_1 b(x), \Omega\big).$$

Let $\alpha > \max\{\lambda, \|u_0\|_\infty\}$. By well-known properties of the logistic equations, we know that the unique solution $u^0(x,t)$ of the following problem

$$\begin{cases} u_t - \Delta u = \lambda u - u^2 - \xi_1 b(x)u, & x \in \Omega,\ t > 1, \\ \partial_\nu u = 0, \quad x \in \partial\Omega,\ t > 1, & u(x,1) = \alpha, \quad x \in \Omega, \end{cases} \tag{3.20}$$

converges to 0 uniformly in $\overline{\Omega}$ as $t \to \infty$. Moreover, by the comparison principle, $u^0(x,t) < \alpha$ for all $t > 1$.

Next we define $\tilde{u}^0(x,t)$ such that it is continuous on $\overline{\Omega} \times (0,\infty)$ and $\tilde{u}^0 = \alpha$ for $t \in [0,1]$, $\tilde{u}^0 \geqslant u^0$ for $t \in [1,2]$, $\tilde{u}^0 = u^0$ for $t > 2$. Let v^0 denote the unique solution to

$$\begin{cases} v_t - \Delta v = \mu v - v^2 - d\tilde{u}^0(x,t)v, & x \in \Omega_1,\ t > 0, \\ \partial_\nu v = 0, \quad x \in \partial\Omega_1,\ t > 0, & v(x,0) = v_0, \quad x \in \Omega_1. \end{cases} \tag{3.21}$$

Then it is easily shown that $v^0(x,t) > 0$ for $t > 0$ and $v^0(x,t) \to \mu$ as $t \to \infty$ uniformly in $\overline{\Omega}_1$.

We also need the following auxiliary problem

$$\begin{cases} v_t - \Delta v = \mu v - v^2 - d\alpha v, & x \in \Omega_1,\ t > 0, \\ \partial_\nu v = 0, \quad x \in \partial\Omega_1,\ t > 0, & v(x,0) = v_0, \quad x \in \Omega_1. \end{cases} \tag{3.22}$$

Let v^1 be the unique solution of (3.22). By the strong maximum principle, $v^1(x,t) > 0$ for all $t > 0$ and $x \in \overline{\Omega}_1$. For any $\eta > \xi_1 / \min_{\overline{\Omega}_1} v^1(\cdot,1)$, we can find $t_\eta > 1$ such that $v^1(x,t) > \xi_1/\eta$ for $t \in [1,t_\eta]$ and all $x \in \overline{\Omega}_1$; moreover, we can choose t_η so that $t_\eta \to \infty$ as $\eta \to \infty$.

Let (u,v) be the unique solution of (1.1) with initial data (u_0,v_0). Then it is easily seen that $u < \alpha$ for all $t > 0$ and hence $v(x,t) > v^1(x,t)$ for all $t > 0$ and $x \in \overline{\Omega}_1$. We claim that $u(x,t) < \tilde{u}^0(x,t)$ for $x \in \overline{\Omega}$ and $t \in [0,t_\eta]$. Indeed, since $u < \alpha$ for $t \geqslant 0$, we obviously have $u < \tilde{u}^0$ for $t \in [0,1]$. For $t \in [1,t_\eta)$, from $v > v^1 \geqslant \xi_1/\eta$ we deduce

$$u_t - \Delta u \leqslant \lambda u - u^2 - \xi_1 b(x)u \quad \text{in } \Omega,$$

with strict inequality over Ω_1. Since $u(x,1) < \alpha$, the comparison principle yields $u(x,t) < u^0(x,t) \leqslant \tilde{u}^0(x,t)$ for $t \in [1,t_\eta]$ and $x \in \overline{\Omega}$.

Let us now fix $T > 2$ such that $v^0(x,t) > \mu/2$ for $t \geqslant T$ and $x \in \overline{\Omega}_1$. Then choose $\eta_0 > 0$ large so that $t_\eta > T$ and $\mu/2 > \xi_1/\eta$ for $\eta \geqslant \eta_0$. We claim that whenever $\eta \geqslant \eta_0$, $u(x,t) < \tilde{u}^0(x,t)$ for all $t > 0$ and $x \in \overline{\Omega}$. Otherwise, for some fixed $\eta \geqslant \eta_0$, we can find $t^* > t_\eta$ and $x^* \in \overline{\Omega}$ such that

$$u(x,t) < \tilde{u}^0(x,t) \quad \text{for } x \in \overline{\Omega},\ t \in \big[0,t^*\big), \qquad u\big(x^*,t^*\big) = \tilde{u}^0\big(x^*,t^*\big).$$

We show next that this is impossible. Firstly, the above inequality implies $v(x,t) \geqslant v^0(x,t)$ for $t \in [0,t^*]$. Since $v^0(x,t) > \mu/2$, for $t \in [T,t^*]$, by continuity, we can find $\delta > 0$ small so that $v(t,x) > \mu/2$ for $t \in [T,t^*+\delta]$. It follows that, for $t \in [T,t^*+\delta]$,

$$u_t - \Delta u \leqslant \lambda u - u^2 - \eta(\mu/2)b(x)u \leqslant \lambda u - u^2 - \xi_1 b(x)u \quad \text{in } \Omega,$$

with strict inequalities in Ω_1. Since $u(x,T) < \tilde{u}^0(x,T) = u^0(x,T)$, the strong maximum principle implies that $u(x,t) < u^0(x,t)$ for $t \in [T,t^*+\delta)$, which is a contradiction to our definition of t^*. Therefore we have $0 < u(x,t) < \tilde{u}^0(x,t)$ for all $t > 0$. Since $\tilde{u}^0(x,t) = u^0(x,t)$ for $t > 2$

and $u^0(x,t) \to 0$ uniformly for $x \in \overline{\Omega}$ as $t \to \infty$, we easily deduce $u(x,t) \to 0$ as $t \to \infty$, uniformly for $x \in \overline{\Omega}$. This implies, by the equation for $v(x,t)$, that $v(x,t) \to \mu$ uniformly in $x \in \overline{\Omega}_1$ as $t \to \infty$. $\square$

Remark 3.14. By Proposition 2.3, it is easily seen that if $(u_0, v_0) \in \mathcal{B}_\eta$ and if $(\phi, \psi) \in K$ satisfies $0 \leqslant \phi \leqslant u_0$, $\psi \geqslant v_0$, then $(\phi, \psi) \in \mathcal{B}_\eta$.

3.4. *Case 4: $\lambda < \lambda_1^D(\Omega_0)$, $\mu > d\lambda$*

In this case, $(\lambda, 0)$ is linearly unstable, $(0, \mu)$ is linearly unstable for $\eta < \eta_0$ and is linearly stable for $\eta > \eta_0$.

Theorem 3.15. *There exists $\eta^* \geqslant \eta_0$ such that* (1.2) *has a stable positive solution when $\eta \in (0, \eta^*)$, and it has no positive solution when $\eta > \eta^*$. Moreover, when $\eta > \eta^*$, $(0, \mu)$ attracts all the positive solutions of* (1.1).

Proof. This is similar to the classical case where no protection zone is present. For the conclusions about the steady-state solutions of (1.1), as before, we can transform (1.2) into an equivalent fixed point problem for some completely continuous operator A which maps the order interval $[(\lambda, 0), (0, \mu)]_{\tilde{K}}$ into itself and is order-preserving. If $\eta \in (0, \eta_0)$, then both $(\lambda, 0)$ and $(0, \mu)$ are linearly unstable, and hence it follows from [8] that A has a stable fixed point (u, v) in $[(\lambda, 0), (0, \mu)]_{\tilde{K}} \setminus \{(\lambda, 0), (0, \mu)\}$. Since $(0, 0)$ is linearly unstable, one easily checks that (u, v) is a stable positive solution of (1.2).

On the other hand, if (u, v) is a positive solution of (1.2), then $0 < u < \lambda$ and hence $-\Delta v > \mu v - v^2 - d\lambda v$, which implies that $v > \mu - d\lambda > 0$. Therefore

$$\lambda = \lambda_1^N\big(u + \eta b(x) v, \Omega\big) > \lambda_1^N\big(\eta(\mu - d\lambda) b(x), \Omega\big).$$

This implies that $\eta(\mu - d\lambda) < \xi_0$, i.e., $\eta < \xi_0/(\mu - d\lambda)$.

Let $\eta^* := \sup\{\eta > 0\colon$ (1.2) has a positive solution$\}$. Then our above discussions imply $\eta_0 \leqslant \eta^* \leqslant \xi_0/(\mu - d\lambda)$. If $\eta^* > \eta_0$, we show that (1.2) has a stable positive solution for each $\eta \in (0, \eta^*)$. (We already proved that this holds when $\eta^* = \eta_0$.) Firstly we claim that (1.2) has a positive solution (u^*, v^*) when $\eta = \eta^*$. Indeed, by definition, we can find $\eta_n \leqslant \eta^*$ such that $\eta_n \to \eta^*$ and (1.2) has a positive solution (u_n, v_n) when $\eta = \eta_n$. Since $0 < u_n < \lambda$ and $0 < v_n < \mu$, from the equations for u_n and v_n we find that they are bounded in $W^{2,p}(\Omega)$ and $W^{2,p}(\Omega_1)$ ($\forall p > 1$), respectively. Hence by passing to a subsequence we may assume that $u_n \to u^*$ and $v_n \to v^*$ weakly in the $W^{2,p}$ norm and strongly in the C^1 norm. It is then easily seen that (u^*, v^*) is a nonnegative solution of (1.2) with $\eta = \eta^*$. We show that it is a positive solution. Otherwise we have either $(u^*, v^*) = (\lambda, 0)$, or $(u^*, v^*) = (0, \mu)$, or $(u^*, v^*) = (0, 0)$. Since we have

$$\lambda = \lambda_1^N\big(u_n + \eta_n b(x) v_n, \Omega\big), \qquad \mu = \lambda_1^N(v_n + d u_n, \Omega_1),$$

it is easy to check that each of the three possibilities leads to a contradiction. Therefore (u^*, v^*) is a positive solution of (1.2) with $\eta = \eta^*$. Now for each $\eta \in (0, \eta^*)$, we have

$$\begin{cases} -\Delta u^* \leqslant (\not\equiv)\, \lambda u^* - \big(u^*\big)^2 - \eta b(x) u^* v^*, & x \in \Omega, \\ -\Delta v^* = \mu v^* - \big(v^*\big)^2 - d u^* v^*, & x \in \Omega_1. \end{cases}$$

It follows that $A(u^*, v^*) >_{\tilde{K}} (u^*, v^*)$. Therefore A maps the order interval $[(\lambda, 0), (u^*, v^*)]_{\tilde{K}}$ into itself. Since $(\lambda, 0)$ is linearly unstable, by [8], A has a stable fixed point in $[(\lambda, 0), (u^*, v^*)]_{\tilde{K}} \setminus \{(\lambda, 0)\}$, i.e., (1.2) has a stable positive solution.

It remains to show that for $\eta > \eta^*$, $(0, \mu)$ attracts all the positive solutions of (1.1). Since $\mu > d\lambda$, we can find $\delta_1 > 0$ small so that $\mu > d(\lambda + \delta_1)$. Let (u, v) be a positive solution of (1.1). Then $u_t - \Delta u \leqslant \lambda u - u^2$ and hence $\limsup_{t\to\infty} u(x, t) \leqslant \lambda$. Therefore there exists $T > 0$ such that $u(x, t) < \lambda + \delta_1$ for $t \geqslant T$ and $x \in \Omega$. We now choose $\delta_2 > 0$ such that $\delta_2 < \min\{\mu - d(\lambda + \delta_1), \min_{\overline{\Omega}_1} v(\cdot, T)\}$. Then it is easily checked that $(u_0, v_0) := (\lambda + \delta_1, \delta_2)$ satisfies

$$\begin{cases} -\Delta u_0 \geqslant \lambda u_0 - u_0^2 - \eta b(x) u_0 v_0, & x \in \Omega, \\ -\Delta v_0 \leqslant \mu v_0 - v_0^2 - d u_0 v_0, & x \in \Omega_1, \\ \partial_\nu u_0 = 0, \quad x \in \partial\Omega, & \partial_\nu v_0 = 0, \quad x \in \partial\Omega_1. \end{cases}$$

By Proposition 2.4, we find that the unique solution (u^0, v^0) of (1.1) with initial data (u_0, v_0) satisfies $(u^0, v^0) \to (U, V)$ as $t \to \infty$, where (U, V) is a nonnegative solution of (1.2) and $U \leqslant u_0$, $V \geqslant v_0$. Since $(0, \mu)$ is the only nonnegative solution of (1.2) with $\eta > \eta^*$ having these properties, we must have $(U, V) = (0, \mu)$.

By our choice of T, we have $u(x, T) < u_0(x)$ and $v(x, T) > v_0(x)$. Now we apply Proposition 2.3 and deduce $u(x, T + t) \leqslant u^0(x, t)$, $v(x, T + t) \geqslant v^0(x, t)$ for $t > 0$. Hence $u(x, t) \to 0$ as $t \to \infty$ and $\liminf_{t\to\infty} v(x, t) \geqslant \mu$. Since $v_t - \Delta v \leqslant \mu v - v^2$, one easily sees that $\limsup_{t\to\infty} v(x, t) \leqslant \mu$. Hence $v(x, t) \to \mu$ as $t \to \infty$. The proof is complete. □

Remark 3.16. It can be proved that when $\eta^* > \eta_0$, (1.2) has at least two positive solutions for $\eta \in (\eta_0, \eta^*)$. One could combine the upper and lower solution method and a bifurcation argument with η as the bifurcation parameter to prove this (see [18] for a related situation).

Remark 3.17. We could make use of an alternative bifurcation argument with μ as the bifurcation parameter, such as in [19], to give a bifurcation picture of the positive solutions of (1.2) for each of the four cases discussed above. For example, when $\lambda > \lambda_1^D(\Omega_0)$, one can take μ as the bifurcation parameter to show that a global branch of positive solutions bifurcates from $\{(\mu, \lambda, 0)\colon \mu \in \mathbf{R}^1\}$ at $\mu = d\lambda$. It is an interesting question to study how this global bifurcation branch becomes unbounded, and how the results in this paper relate to this global bifurcation branch. There is a similar global bifurcation branch for the case $\lambda < \lambda_1^D(\Omega_0)$, and we may ask the same questions.

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